1.2 FUNCTIONS AND THEIR PROPERTIES

What you'll learn about

- Function Definition and Notation
- Domain and Range
- Continuity
- Increasing and Decreasing Functions
- Boundedness
- Local and Absolute Extrema
- Symmetry
- Asymptotes
- End Behavior

... and why

Functions and graphs form the basis for understanding the mathematics and applications you will see both in your work place and in coursework in college.

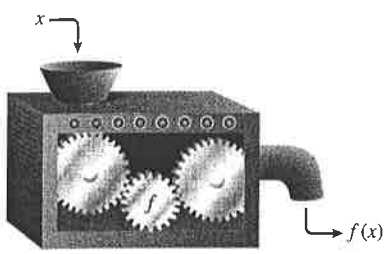


FIGURE 1.9 A "machine" diagram for a function.

A Bit of History

The word function in its mathematical sense is generally attributed to Gottfried Leibniz (1646–1716), one of the pioneers in the methods of calculus. His attention to clarity of notation is one of his greatest contributions to scientific progress, which is why we still use his notation in calculus courses today. Ironically, it was not Leibniz but Leonhard Euler (1707–1783) who introduced the familiar notation $f(x)$.

Function Definition and Notation

Mathematics and its applications abound with examples of formulas by which quantitative variables are related to each other. The language and notation of functions is ideal for that purpose. A function is actually a simple concept; if it were not, history would have replaced it with a simpler one by now. Here is the definition.

> **Definition Function, Domain, and Range**
>
> A function from a set D to a set R is a rule that assigns to every element in D a unique element in R. The set D of all input values is the **domain** of the function, and the set R of all output values is the **range** of the function.

There are many ways to look at functions. One of the most intuitively helpful is the "machine" concept (Figure 1.9), in which values of the domain (x) are fed into the machine (the function f) to produce range values (y). To indicate that y comes from the function acting on x, we use Euler's elegant **function notation** $y = f(x)$ (which we read as "**y equals f of x**" or "**the value of f at x**"). Here x is the **independent variable** and y is the **dependent variable**.

A function can also be viewed as a **mapping** of the elements of the domain onto the elements of the range. Figure 1.10a shows a function that maps elements from the domain X onto elements of the range Y. Figure 1.10b shows another such mapping, but *this one is not a function*, since the rule does not assign the element x_1 to a *unique* element of Y.

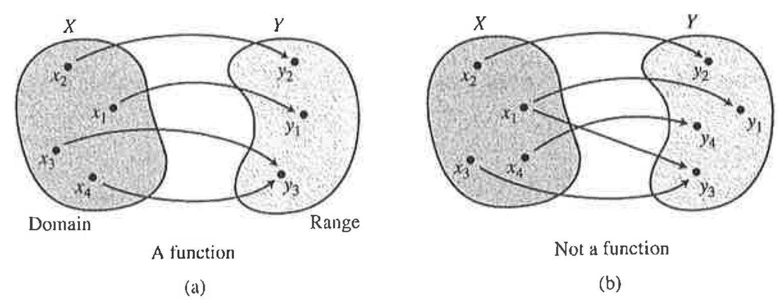


FIGURE 1.10 The diagram in (a) depicts a mapping from X to Y that is a function. The diagram in (b) depicts a mapping from X to Y that is not a function.

This uniqueness of the range value is very important to us as we study function behavior. Knowing that $f(2) = 8$ tells us something about f, and that understanding would be contradicted if we were to discover later that $f(2) = 4$. That is why you will never see a function defined by an ambiguous formula like $f(x) = 3x \pm 2$.

EXAMPLE 1 Defining a function

Does the formula $y = x^2$ define y as a function of x?

SOLUTION

Yes, y is a function of x. In fact, we can write the formula in function notation: $f(x) = x^2$. When a number x is substituted into the function, the square of x will be the output, and there is no ambiguity about what the square of x is. Now try Exercise 3.

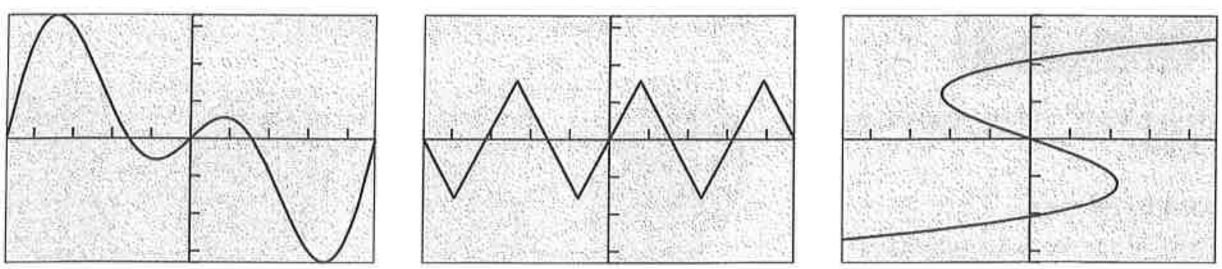
Another useful way to look at functions is graphically. The **graph of the function** $y = f(x)$ is the set of all points $(x, f(x))$, x in the domain of f. We match domain values along the x-axis with their range values along the y-axis to get the ordered pairs that yield the graph of $y = f(x)$.

EXAMPLE 2 Seeing a function graphically

Of the three graphs shown in Figure 1.11, which is *not* the graph of a function? How can you tell?

SOLUTION

The graph in (c) is not the graph of a function. There are three points on the graph with x-coordinate 0, so the graph does not assign a *unique* value to 0. (Indeed, we can see that there are plenty of numbers between –2 and 2 to which the graph assigns multiple values.) The other two graphs do not have a comparable problem because no vertical line intersects either of the other graphs in more than one point. Graphs that pass this *vertical line test* are the graphs of functions. Now try Exercise 5.



[–4.7, 4.7] by [–3.3, 3.3]
(a)

[–4.7, 4.7] by [–3.3, 3.3]
(b)

[–4.7, 4.7] by [–3.3, 3.3]
(c)

FIGURE 1.11 One of these is not the graph of a function. (Example 2)

Vertical Line Test

A graph (set of points (x, y)) in the xy-plane defines y as a function of x if and only if no vertical line intersects the graph in more than one point.

Domain and Range

We will usually define functions algebraically, giving the rule explicitly in terms of the domain variable. The rule, however, does not tell the complete story without some consideration of what the domain actually is.

For example, we can define the volume of a sphere as a function of its radius by the formula

$$V(r) = \frac{4}{3}\pi r^3 \text{ (Note that this is "}V\text{ of }r\text{"—not "}V \cdot r\text{").}$$

This *formula* is defined for all real numbers, but the volume *function* is not defined for negative r values. So, if our intention were to study the volume function, we would restrict the domain to be all $r \geq 0$.

Agreement

Unless we are dealing with a model (like volume) that necessitates a restricted domain, we will assume that the domain of a function defined by an algebraic expression is the same as the domain of the algebraic expression, the **implied domain**. For models, we will use a domain that fits the situation, the **relevant domain**.

EXAMPLE 3 Finding the domain of a function

Find the domain of each of these functions:

(a) $f(x) = \sqrt{x + 3}$

(b) $g(x) = \dfrac{\sqrt{x}}{x - 5}$

(c) $A(s) = (\sqrt{3}/4)s^2$, where $A(s)$ is the area of an equilateral triangle with sides of length s.

SOLUTION

Solve Algebraically

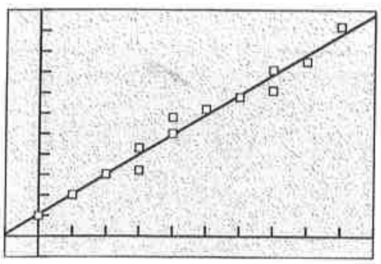
(a) The expression under a radical may not be negative. We set $x + 3 \geq 0$ and solve to find $x \geq -3$. The domain of f is the interval $[-3, \infty)$.

(b) The expression under a radical may not be negative; therefore $x \geq 0$. Also, the denominator of a fraction may not be zero; therefore $x \neq 5$. The domain of g is the interval $[0, \infty)$ with the number 5 removed, which we can write as the *union* of two intervals: $[0, 5) \cup (5, \infty)$.

(c) The algebraic expression has domain all real numbers, but the behavior being modeled restricts s from being negative. The domain of A is the interval $[0, \infty)$.

WHAT ABOUT DATA?

When moving from a numerical model to an algebraic model we will often use a function to approximate data pairs that by themselves violate our definition. In Figure 1.12 we can see that several pairs of data points fail the vertical line test, and yet the linear function approximates the data quite well.



$[-1, 10]$ by $[-1, 11]$

FIGURE 1.12 The data points fail the vertical line test but are nicely approximated by a linear function.

NOTE

The symbol "∪" is read "union." It means that the elements of the two sets are combined to form one set.

Support Graphically

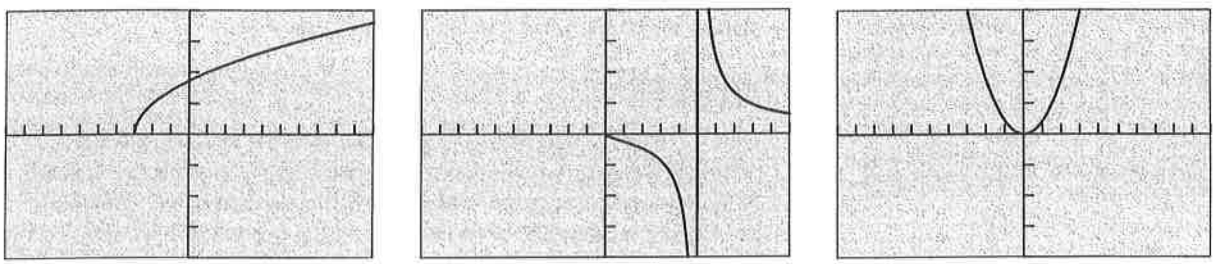
We can support our answers in (a) and (b) graphically, as the calculator should not plot points where the function is undefined.

(a) Notice that the graph of $y = \sqrt{x + 3}$ (Figure 1.13a) shows points only for $x \geq -3$, as expected.

(b) The graph of $y = \sqrt{x}/(x - 5)$ (Figure 1.13b) shows points only for $x \geq 0$, as expected, but shows an unexpected line through the x-axis at $x = 5$. This line, a form of grapher failure described in the previous section, should not be there. Ignoring it, we see that 5, as expected, is not in the domain.

(c) The graph of $y = (\sqrt{3}/4)s^2$ (Figure 1.13c) shows the unrestricted domain of the algebraic expression: all real numbers. The calculator has no way of knowing that s is the length of a side of a triangle.

Now try Exercise 11.



[–10, 10] by [–4, 4]
(a)

[–10, 10] by [–4, 4]
(b)

[–10, 10] by [–4, 4]
(c)

FIGURE 1.13 Graphical support of the algebraic solutions in Example 3. The vertical line in (b) should be ignored because it results from grapher failure. The points in (c) with negative x-coordinates should be ignored because the calculator does not know that x is a length (but we do).

FUNCTION NOTATION

A grapher typically does not use function notation. So the function $f(x) = x^2 + 1$ is entered as $y_1 = x^2 + 1$. On some graphers you can evaluate f at $x = 3$ by entering $y_1(3)$ on the home screen. On the other hand, on other graphers $y_1(3)$ means $y_1 * 3$. (Consult Technology Ancillary.)

Finding the range of a function algebraically is often much harder than finding the domain, although graphically the things we look for are similar: To find the *domain* we look for all *x-coordinates* that correspond to points on the graph, and to find the *range* we look for all *y-coordinates* that correspond to points on the graph. A good approach is to use graphical and algebraic approaches simultaneously, as we show in Example 4.

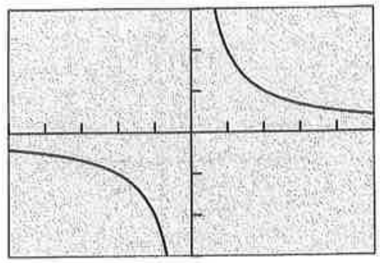
EXAMPLE 4 Finding the range of a function

Find the range of the function $f(x) = \dfrac{2}{x}$.

SOLUTION

Solve Graphically

The graph of $y = \dfrac{2}{x}$ is shown in Figure 1.14.



[–5, 5] by [–3, 3]

FIGURE 1.14 The graph of $y = 2/x$. Is $y = 0$ in the range?

It appears that $x = 0$ is not in the domain (as expected, because a denominator can not be zero). It also appears that the range consists of all real numbers except 0.

Confirm Algebraically

We confirm that 0 is not in the range by trying to solve $2/x = 0$:

$$\frac{2}{x} = 0$$
$$2 = 0 \cdot x$$
$$2 = 0$$

Since the equation $2 = 0$ is never true, $2/x = 0$ has no solutions, and so $y = 0$ is not in the range. But how do we know that all other real numbers are in the range? We let k be any other real number and try to solve $2/x = k$:

$$\frac{2}{x} = k$$
$$2 = k \cdot x$$
$$x = \frac{2}{k}$$

As you can see, there was no problem finding an x this time, so 0 is the only number not in the range of f. We write the range $(-\infty, 0) \cup (0, \infty)$.

Now try Exercise 17.

You can see that this is considerably more involved than finding a domain, but we are hampered at this point by not having many tools with which to analyze function behavior. We will revisit the problem of finding ranges in Exercise 86, after having developed the tools that will simplify the analysis.

Continuity

One of the most important properties of the majority of functions that model real-world behavior is that they are *continuous*. Graphically speaking, a function is continuous at a point if the graph does not come apart at that point. We can illustrate the concept with a few graphs (Figure 1.15):

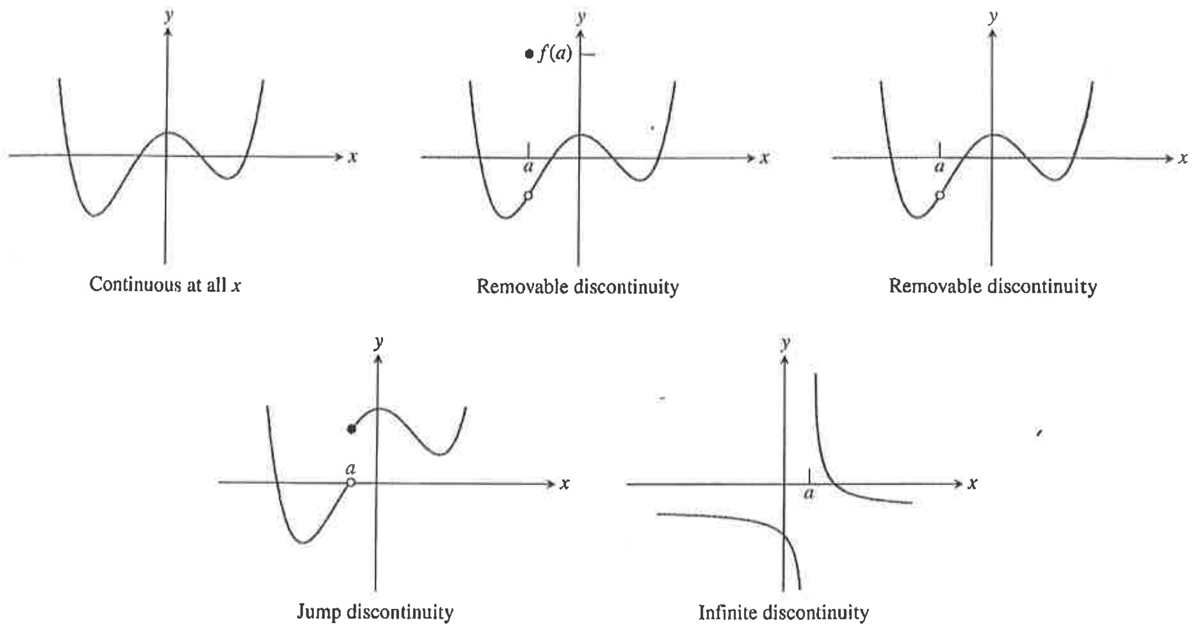
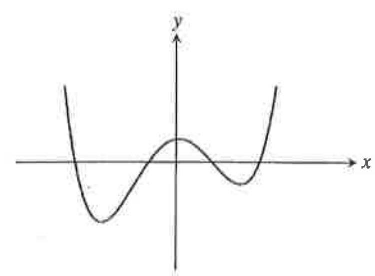


FIGURE 1.15 Some points of discontinuity.

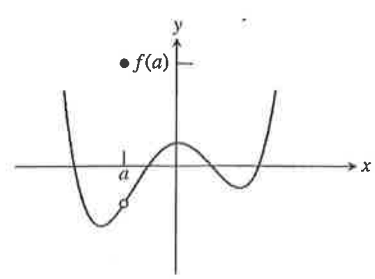
Let's look at these cases individually.

This graph is continuous everywhere. Notice that the graph has no breaks. This means that if we are studying the behavior of the function f for x values close to any particular real number a, we can be assured that the $f(x)$ values will be close to $f(a)$.



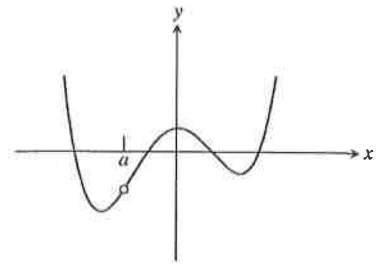
Continuous at all x

This graph is continuous everywhere except for the "hole" at $x = a$. If we are studying the behavior of this function f for x values close to a, we can *not* be assured that the $f(x)$ values will be close to $f(a)$. In this case, $f(x)$ is smaller than $f(a)$ for x near a. This is called a **removable discontinuity** because it can be patched by redefining $f(a)$ so as to plug the hole.



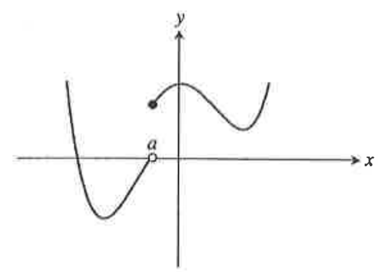
Removable discontinuity

This graph also has a **removable discontinuity** at $x = a$. If we are studying the behavior of this function f for x values close to a, we are still not assured that the $f(x)$ values will be close to $f(a)$, because in this case $f(a)$ doesn't even exist. It is removable because we could define $f(a)$ in such a way as to plug the hole and make f continuous at a.



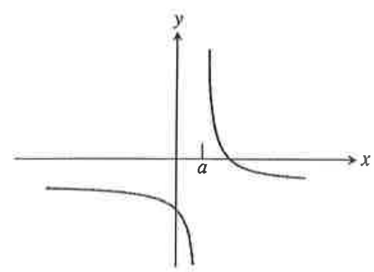
Removable discontinuity

Here is a discontinuity that is not removable. It is a **jump discontinuity** because there is more than just a hole at $x = a$; there is a *jump* in function values that makes the gap impossible to plug with a single point $(a, f(a))$, no matter how we try to redefine $f(a)$.



Jump discontinuity

This is a function with an **infinite discontinuity** at $x = a$. It is definitely not removable.



Infinite discontinuity

The simple geometric concept of an unbroken graph at a point is one of those visual notions that is extremely difficult to communicate accurately in the language of algebra. The key concept from the pictures seems to be that we want the point $(x, f(x))$ to slide smoothly onto the point $(a, f(a))$ without missing it, from either direction. This merely requires that $f(x)$ approach $f(a)$ as a *limit* as x approaches a. A function f is **continuous at** $x = a$ if $\lim_{x \to a} f(x) = f(a)$. A function f is **discontinuous at** $x = a$ if it is not continuous at $x = a$.

CHOOSING VIEWING WINDOWS

Most viewing windows will show a vertical line for the function in Figure 1.16. It is sometimes possible to choose a viewing window in which the vertical line does not appear, as we did in Figure 1.16.

EXAMPLE 5 Identifying points of discontinuity

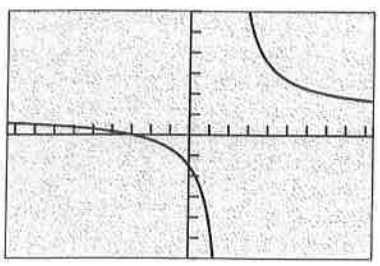
Judging from the graphs, which of the following figures shows functions that are discontinuous at $x = 2$? Are any of the discontinuities removable?

SOLUTION

Figure 1.16 shows a function that is undefined at $x = 2$ and hence not continuous there. The discontinuity at $x = 2$ is not removable.

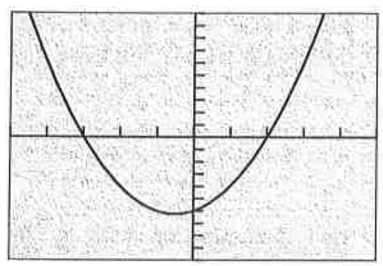
The function graphed in Figure 1.17 is a quadratic polynomial whose graph is a parabola, a graph that has no breaks because its domain includes all real numbers. It is continuous for all x.

The function graphed in Figure 1.18 is not defined at $x = 2$ and so can not be continuous there. The graph looks like the graph of the line $y = x + 2$, except that there is a hole where the point $(2, 4)$ should be. This is a removable discontinuity. Now try Exercise 21.



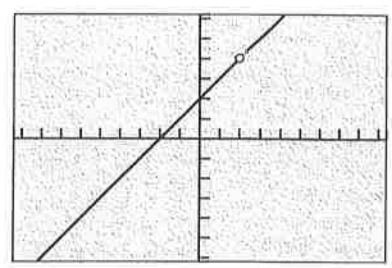
$[-9.4, 9.4]$ by $[-6, 6]$

FIGURE 1.16 $f(x) = \dfrac{x+3}{x-2}$



$[-5, 5]$ by $[-10, 10]$

FIGURE 1.17 $g(x) = (x + 3)(x - 2)$

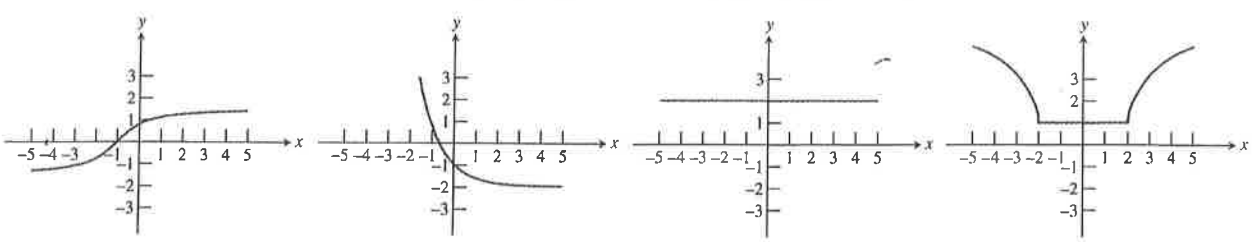


$[-9.4, 9.4]$ by $[-6.2, 6.2]$

FIGURE 1.18 $h(x) = \dfrac{x^2 - 4}{x - 2}$

Increasing and Decreasing Functions

Another function concept that is easy to understand graphically is the property of being increasing, decreasing, or constant on an interval. We illustrate the concept with a few graphs (Figure 1.19):



Increasing

Decreasing

Constant

Decreasing on $(-\infty, -2]$
Constant on $[-2, 2]$
Increasing on $[2, \infty)$

FIGURE 1.19 Examples of increasing, decreasing, or constant on an interval.

Once again the idea is easy to communicate graphically, but how can we identify these properties of functions algebraically? Exploration 1 will help to set the stage for the algebraic definition.

EXPLORATION 1 Increasing, Decreasing, and Constant Data

1. Of the three tables of numerical data below, which would be modeled by a function that is (a) increasing, (b) decreasing, (c) constant?

X	Y1
−2	12
−1	12
0	12
1	12
3	12
7	12

X	Y2
−2	3
−1	1
0	0
1	−2
3	−6
7	−12

X	Y3
−2	−5
−1	−3
0	−1
1	1
3	4
7	10

2. Make a list of $\Delta Y1$, the *change* in Y1 values as you move down the list. As you move from $Y1 = a$ to $Y1 = b$, the change is $\Delta Y1 = b - a$. Do the same for the values of Y2 and Y3.

X moves from	ΔX	ΔY1
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	ΔY2
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

X moves from	ΔX	ΔY3
−2 to −1	1	
−1 to 0	1	
0 to 1	1	
1 to 3	2	
3 to 7	4	

3. What is true about the quotients $\Delta Y/\Delta X$ for an increasing function? For a decreasing function? For a constant function?

4. Where else have you seen the quotient $\Delta Y/\Delta X$? Does this reinforce your answers in part 3?

ΔList on a Calculator

Your calculator might be able to help you with the numbers in Exploration 1. Some calculators have a "ΔList" operation that will calculate the changes as you move down a list. For example, the command "ΔList (L1) → L3" will store the differences from L1 into L3. Note that ΔList (L1) is always one entry shorter than L1 itself.

Your analysis of the quotients $\Delta Y/\Delta X$ in the exploration should help you to understand the following definition.

Definition Increasing, Decreasing, and Constant Function on an Interval

A function f is **increasing** on an interval if, for any two points in the interval, a positive change in x results in a positive change in $f(x)$.

A function f is **decreasing** on an interval if, for any two points in the interval, a positive change in x results in a negative change in $f(x)$.

A function f is **constant** on an interval if, for any two points in the interval, a positive change in x results in a zero change in $f(x)$.

EXAMPLE 6 Analyzing a function for increasing-decreasing behavior

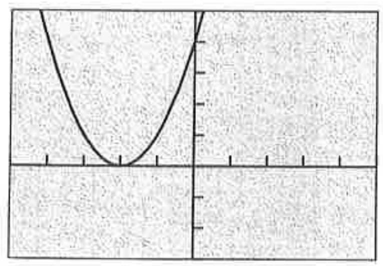
For each function, tell the intervals on which it is increasing and the intervals on which it is decreasing.

(a) $f(x) = (x + 2)^2$ **(b)** $g(x) = \dfrac{x^2}{x^2 - 1}$

SOLUTION

Solve Graphically

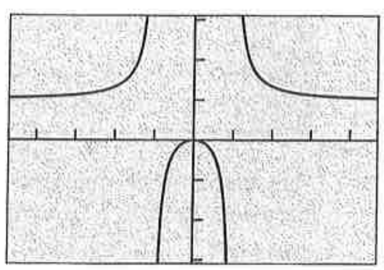
(a) We see from the graph in Figure 1.20 that f is decreasing on $(-\infty, -2]$ and increasing on $[-2, \infty)$. (Notice that we include -2 in both intervals. Don't worry that this sets up some contradiction about what happens *at* -2, because we only talk about functions increasing or decreasing on *intervals*, and -2 is not an interval.)



[–5, 5] by [–3, 5]

FIGURE 1.20 The function $f(x) = (x + 2)^2$ decreases on $(-\infty, -2]$ and increases on $[-2, \infty)$. (Example 6

(b) We see from the graph in Figure 1.21 that g is increasing on $(-\infty, -1)$, increasing again on $(-1, 0]$, decreasing on $[0, 1)$, and decreasing again on $(1, \infty)$.



[–4.7, 4.7] by [–3.1, 3.1]

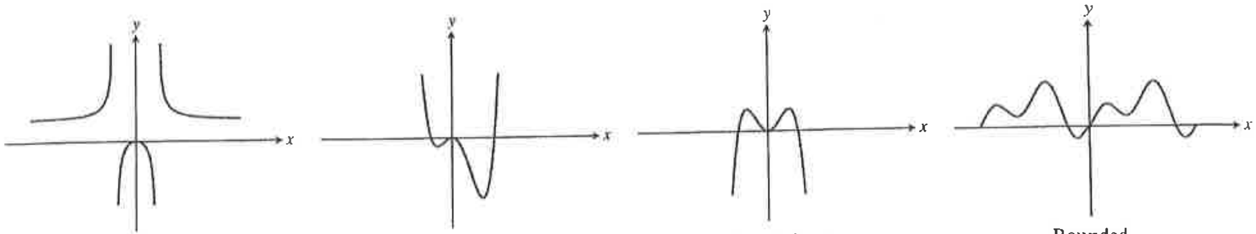
FIGURE 1.21 The function $g(x) = x^2/(x^2 - 1)$ increases on $(-\infty, -1)$ and $(-1, 0]$; the function decreases on $[0, 1)$ and $(1, \infty)$. (Example 6)

Now try Exercise 33.

You may have noticed that we are making some assumptions about the graphs. How do we know that they don't turn around somewhere off the screen? We will develop some ways to answer that question later in the book, but the most powerful methods will await you when you study calculus.

Boundedness

The concept of *boundedness* is fairly simple to understand both graphically and algebraically. We will move directly to the algebraic definition after motivating the concept with some typical graphs (Figure 1.22).



Not bounded above
Not bounded below

Not bounded above
Bounded below

Bounded above
Not bounded below

Bounded

FIGURE 1.22 Some examples of graphs bounded and not bounded above and below.

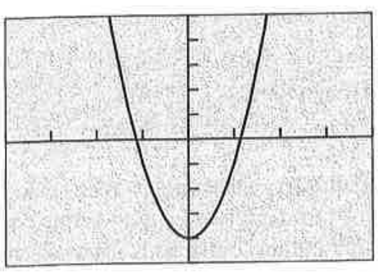
Definition Lower Bound, Upper Bound, and Bounded

A function f is **bounded below** if there is some number b that is less than or equal to every number in the range of f. Any such number b is called a **lower bound** of f.

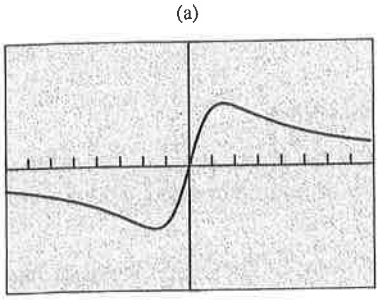
A function f is **bounded above** if there is some number B that is greater than or equal to every number in the range of f. Any such number B is called an **upper bound** of f.

A function f is **bounded** if it is bounded both above and below.

We can extend the above definition to the idea of **bounded on an interval** by restricting the domain of consideration in each part of the definition to the interval we wish to consider. For example, the function $f(x) = 1/x$ is bounded above on the interval $(-\infty, 0)$ and bounded below on the interval $(0, \infty)$.



[–4, 4] by [–5, 5]
(a)



[–8, 8] by [–1, 1]
(b)

FIGURE 1.23 The graphs for Example 7. Which are bounded where?

EXAMPLE 7 Checking boundedness

Identify each of these functions as bounded below, bounded above, or bounded.

(a) $w(x) = 3x^2 - 4$ **(b)** $p(x) = \dfrac{x}{1 + x^2}$

SOLUTION

Solve Graphically

The two graphs are shown in Figure 1.23. It appears that w is bounded below, and p is bounded.

Confirm Algebraically

We can confirm that w is bounded below by finding a lower bound, as follows:

$$x^2 \geq 0 \quad x^2 \text{ is nonnegative}$$
$$3x^2 \geq 0 \quad \text{Multiply by 3.}$$
$$3x^2 - 4 \geq 0 - 4 \quad \text{Subtract 4.}$$
$$3x^2 - 4 \geq -4$$

Thus, -4 is a lower bound for $w(x) = 3x^2 - 4$.

We leave the verification that p is bounded as an exercise (Exercise 77).

Now try Exercise 37.

Local and Absolute Extrema

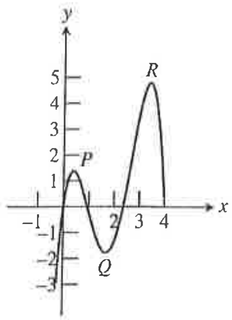


FIGURE 1.24 The graph suggests that f has a local maximum at P, a local minimum at Q, and a local maximum at R.

Many graphs are characterized by peaks and valleys where they change from increasing to decreasing and vice versa. The extreme values of the function (or *local extrema*) can be characterized as either *local maxima* or *local minima*. The distinction can be easily seen graphically. Figure 1.24 shows a graph with three local extrema: local maxima at points P and R and a local minimum at Q.

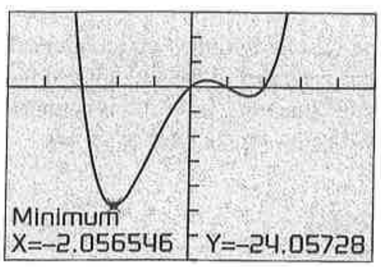
This is another function concept that is easier to see graphically than to describe algebraically. Notice that a local maximum does not have to be *the* maximum value of a function; it only needs to be the maximum value of the function on *some* tiny interval.

Definition Local and Absolute Extrema

A **local maximum** of a function f is a value $f(c)$ that is greater than or equal to all range values of f on some open interval containing c. If $f(c)$ is greater than or equal to all range values of f, then $f(c)$ is the **maximum** (or **absolute maximum**) value of f.

A **local minimum** of a function f is a value $f(c)$ that is less than or equal to all range values of f on some open interval containing c. If $f(c)$ is less than or equal to all range values of f, then $f(c)$ is the **minimum** (or **absolute minimum**)value of f.

Local extrema are also called **relative extrema**.



[–5, 5] by [–35, 15]

FIGURE 1.25 A graph of $y = x^4 - 7x^2 + 6x$. (Example 8)

We have already mentioned that the best method for analyzing increasing and decreasing behavior involves calculus. Not surprisingly, the same is true for local extrema. We will generally be satisfied in this course with approximating local extrema using a graphing calculator, although sometimes an algebraic confirmation will be possible when we learn more about specific functions.

USING A GRAPHER TO FIND LOCAL EXTREMA

Most modern graphers have built in "maximum" and "minimum" finders that identify local extrema by looking for sign changes in Δy. It is not easy to find local extrema by zooming in on them, as the graphs tend to flatten out and hide the very behavior you are looking at. If you use this method, keep narrowing the vertical window to maintain some curve in the graph.

EXAMPLE 8 Identifying local extrema

Decide whether $f(x) = x^4 - 7x^2 + 6x$ has any local maxima or local minima. If so, find each local maximum value or minimum value and the value of x at which each occurs.

SOLUTION The graph of $y = x^4 - 7x^2 + 6x$ (Figure 1.25) suggests that there are two local minimum values and one local maximum value. We use the graphing calculator to approximate local minima as -24.06 (which occurs at $x \approx -2.06$) and -1.77 (which occurs at $x \approx 1.60$). Similarly, we identify the (approximate) local maximum as 1.32 (which occurs at $x \approx 0.46$). Now try Exercise 41.

Symmetry

In the graphical sense, the word "symmetry" in mathematics carries essentially the same meaning as it does in art: The picture (in this case, the graph) "looks the same" when viewed in more than one way. The interesting thing about mathematical symmetry is that it can be characterized numerically and algebraically as well. We will be looking at three particular types of symmetry, each of which can be spotted easily from a graph, a table of values, or an algebraic formula, once you know what to look for. Since it is the connections among the three models (graphical, numerical, and algebraic) that we need to emphasize in this section, we will illustrate the various symmetries in all three ways, side-by-side.

Symmetry with respect to the y-axis
Example: $f(x) = x^2$

Graphically

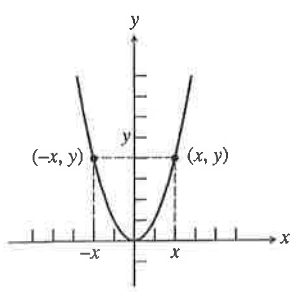


FIGURE 1.26 The graph looks the same to the left of the y-axis as it does to the right of it.

Numerically

x	$f(x)$
−3	9
−2	4
−1	1
1	1
2	4
3	9

Algebraically

For all x in the domain of f,

$$f(-x) = f(x)$$

Functions with this property (e.g., x^n, n even) are **even** functions.

Symmetry with respect to the x-axis
Example: $x = y^2$

Graphically

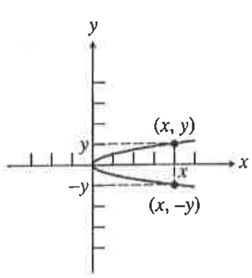


FIGURE 1.27 The graph looks the same above the x-axis as it does below it.

Numerically

x	y
9	−3
4	−2
1	−1
1	1
4	2
9	3

Algebraically

Graphs with this kind of symmetry are not functions (except the zero function), but we can say that $(x, -y)$ is on the graph whenever (x, y) is on the graph.

Symmetry with respect to the origin
Example: $f(x) = x^3$

Graphically

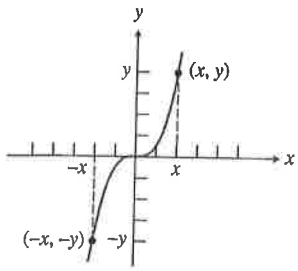


FIGURE 1.28 The graph looks the same upside-down as it does rightside-up.

Numerically

x	y
−3	−27
−2	−8
−1	−1
1	1
2	8
3	−27

Algebraically

For all x in the domain of f,

$$f(-x) = -f(x).$$

Functions with this property (e.g., x^n, n odd) are **odd** functions.

EXAMPLE 9 Checking functions for symmetry

Tell whether each of the following functions is odd, even, or neither.

(a) $f(x) = x^2 - 3$

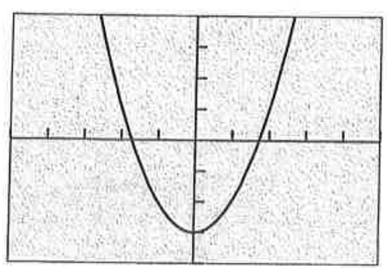
(b) $g(x) = x^2 - 2x - 2$

(c) $h(x) = \dfrac{x^3}{4 - x^2}$

SOLUTION

(a) Solve Graphically

The graphical solution is shown in Figure 1.29.



[−5, 5] by [−4, 4]

FIGURE 1.29 This graph appears to be symmetric with respect to the y-axis, so we conjecture that f is an even function.

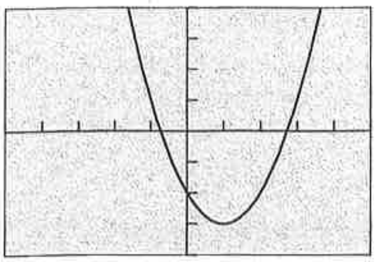
Confirm Algebraically

We need to verify that

$$f(-x) = f(x)$$

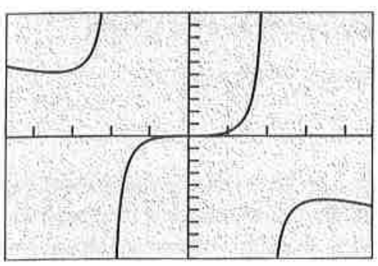
for all x in the domain of f.

$$f(-x) = (-x)^2 - 3 = x^2 - 3$$
$$= f(x)$$



[–5, 5] by [–4, 4]

FIGURE 1.30 This graph does not appear to be symmetric with respect to either the y-axis or the origin, so we conjecture that g is neither even nor odd.



[–4.7, 4.7] by [–10, 10]

FIGURE 1.31 This graph appears to be symmetric with respect to the origin, so we conjecture that h is an odd function.

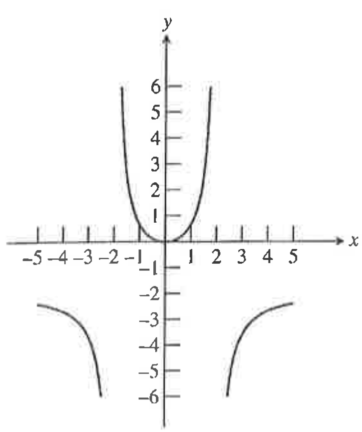


FIGURE 1.32 The graph of $f(x) = 2x^2/(4 - x^2)$ has two vertical asymptotes and one horizontal asymptote.

Since this identity is true for all x, the function f is indeed even.

(b) Solve Graphically

The graphical solution is shown in Figure 1.30.

Confirm Algebraically

We need to verify that

$$g(-x) \neq g(x) \text{ and } g(-x) \neq -g(x).$$

$$\begin{aligned} g(-x) &= (-x)^2 - 2(-x) - 2 \\ &= x^2 + 2x - 2 \end{aligned}$$

$$\begin{aligned} g(x) &= x^2 - 2x - 2 \\ -g(x) &= -x^2 + 2x + 2 \end{aligned}$$

So $g(-x) \neq g(x)$ and $g(-x) \neq -g(x)$.

We conclude that g is neither odd nor even.

(c) Solve Graphically

The graphical solution is shown in Figure 1.31.

Confirm Algebraically

We need to verify that

$$h(-x) = -h(x)$$

for all x in the domain of h.

$$\begin{aligned} h(-x) &= \frac{(-x)^3}{4 - (-x)^2} = \frac{-x^3}{4 - x^2} \\ &= -h(x) \end{aligned}$$

Since this identity is true for all x except ± 2 (which are not in the domain of h), the function h is odd. Now try Exercise 49.

Asymptotes

Consider the graph of the function $f(x) = \dfrac{2x^2}{4 - x^2}$ in Figure 1.32.

The graph appears to flatten out to the right and to the left, getting closer and closer to the horizontal line $y = -2$. We call this line a *horizontal asymptote*. Similarly, the graph appears to flatten out as it goes off the top and bottom of the screen, getting closer and closer to the vertical lines $x = -2$ and $x = 2$. We call these lines *vertical asymptotes*. If we superimpose the asymptotes onto Figure 1.32 as dashed lines, you can see that they form a kind of template that describes the limiting behavior of the graph (Figure 1.33).

Since asymptotes describe the behavior of the graph at its horizontal or vertical extremities, the definition of an asymptote can best be stated with limit notation. In this definition, note that $x \to a^-$ means "x approaches a from the left," while $x \to a^+$ means "x approaches a from the right."

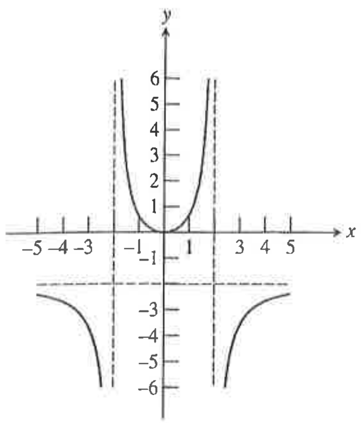
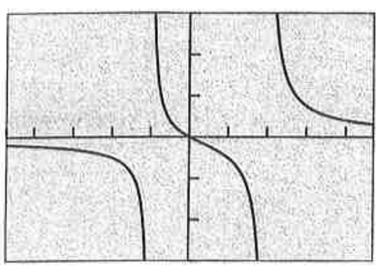


FIGURE 1.33 The graph of $f(x) = 2x^2/(4 - x^2)$ with the asymptotes shown as dashed lines.



[-4.7, 4.7] by [-3, 3]

FIGURE 1.34 The graph of $y = x/(x^2 - x - 2)$ has vertical asymptotes of $x = -1$ and $x = 2$ and a horizontal asymptote of $y = 0$. (Example 10)

Definition Horizontal and Vertical Asymptotes

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of b as x approaches $+\infty$ or $-\infty$.

In limit notation:

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to +\infty} f(x) = b.$$

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of $+\infty$ or $-\infty$ as x approaches a from either direction.

In limit notation:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

EXAMPLE 10 Identifying the asymptotes of a graph

Identify any horizontal or vertical asymptotes of the graph of

$$y = \frac{x}{x^2 - x - 2}.$$

SOLUTION The quotient $x/(x^2 - x - 2) = x/((x + 1)(x - 2))$ is undefined at $x = -1$ and $x = 2$, which makes them likely sites for vertical asymptotes. The graph (Figure 1.34) provides support, showing vertical asymptotes of $x = -1$ and $x = 2$.

For large values of x, the numerator (a large number) is dwarfed by the denominator (a *product* of *two* large numbers), suggesting that $\lim_{x \to \infty} x/((x + 1)(x - 2)) = 0$. This would indicate a horizontal asymptote of $y = 0$. The graph (Figure 1.34) provides support, showing a horizontal asymptote of $y = 0$ as $x \to \infty$. Similar logic suggests that $\lim_{x \to -\infty} x/((x + 1)(x - 2)) = -0 = 0$, indicating the same horizontal asymptote as $x \to -\infty$. Again, the graph provides support for this.

Now try Exercise 57.

End Behavior

A horizontal asymptote gives one kind of end behavior for a function because it shows how the function behaves as it goes off toward either "end" of the x-axis. Not all graphs approach lines, but it is helpful to consider what *does* happen "out there." We illustrate with a few examples.

EXAMPLE 11 Matching functions using end behavior

Match the functions with the graphs in Figure 1.35 by considering end behavior. All graphs are shown in the same viewing window.

(a) $y = \dfrac{3x}{x^2+1}$ **(b)** $y = \dfrac{3x^2}{x^2+1}$ **(c)** $y = \dfrac{3x^3}{x^2+1}$ **(d)** $y = \dfrac{3x^4}{x^2+1}$

SOLUTION

When x is very large, the denominator $x^2 + 1$ in each of these functions is almost the same number as x^2. If we replace $x^2 + 1$ in each denominator by x^2 and then reduce the fractions, we get the simpler functions

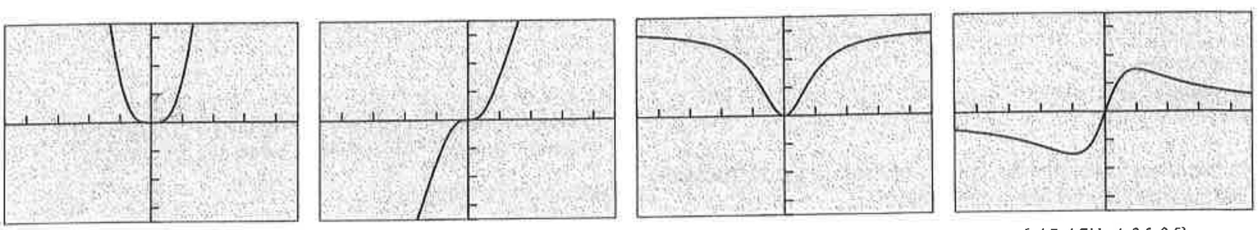
(a) $y = \dfrac{3}{x}$ (close to $y = 0$ for large x) **(b)** $y = 3$ **(c)** $y = 3x$ **(d)** $y = 3x^2$.

So, we look for functions that have end behavior resembling, respectively, the functions

(a) $y = 0$ **(b)** $y = 3$ **(c)** $y = 3x$ **(d)** $y = 3x^2$.

Graph (iv) approaches the line $y = 0$. Graph (iii) approaches the line $y = 3$. Graph (ii) approaches the line $y = 3x$. Graph (i) approaches the parabola $y = 3x^2$. So, (a) matches (iv), (b) matches (iii), (c) matches (ii), and (d) matches (i).

Now try Exercise 65.



[–4.7, 4.7] by [–3.5, 3.5]
(i)

[–4.7, 4.7] by [–3.5, 3.5]
(ii)

[–4.7, 4.7] by [–3.5, 3.5]
(iii)

[–4.7, 4.7] by [–3.5, 3.5]
(iv)

FIGURE 1.35 Match the graphs with the functions in Example 11.

For more complicated functions we are often content with knowing whether the end behavior is bounded or unbounded in either direction.

QUICK REVIEW 1.2

(For help, go to Sections A.3, P.3, and P.5.)

In Exercises 1–4, solve the equation or inequality.

1. $x^2 - 16 = 0$

2. $9 - x^2 = 0$

3. $x - 10 < 0$

4. $5 - x \le 0$

In Exercises 5–10, find all values of x algebraically for which the algebraic expression is *not* defined. Support your answer graphically.

5. $\dfrac{x}{x-16}$

6. $\dfrac{x}{x^2-16}$

7. $\sqrt{x-16}$

8. $\dfrac{\sqrt{x^2+1}}{x^2-1}$

9. $\dfrac{\sqrt{x+2}}{\sqrt{3-x}}$

10. $\dfrac{x^2-2x}{x^2-4}$

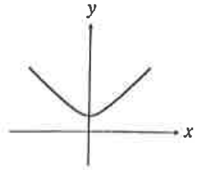
SECTION 1.2 EXERCISES

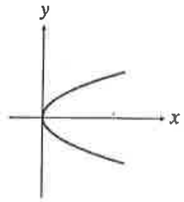
In Exercises 1–4, determine whether the formula determines y as a function of x. If not, explain why not.

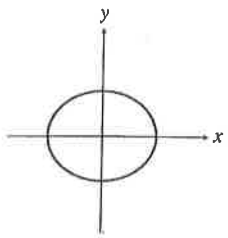
1. $y = \sqrt{x - 4}$ **2.** $y = x^2 \pm 3$

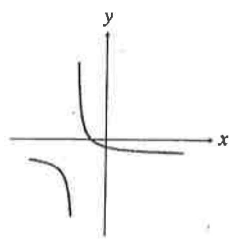
3. $x = 2y^2$ **4.** $x = 12 - y$

In Exercises 5–8, use the vertical line test to determine whether the curve is the graph of a function.

5.


6.


7.


8.


In Exercises 9–16, find the domain of the function algebraically and support your answer graphically.

9. $f(x) = \sqrt{x^2 + 4}$ **10.** $h(x) = \dfrac{5}{x - 3}$

11. $f(x) = \dfrac{3x - 1}{(x + 3)(x - 1)}$ **12.** $f(x) = \dfrac{1}{x} + \dfrac{5}{x - 3}$

13. $g(x) = \dfrac{x}{x^2 - 5x}$ **14.** $h(x) = \dfrac{\sqrt{4 - x^2}}{x - 3}$

15. $h(x) = \dfrac{\sqrt{4 - x}}{(x + 1)(x^2 + 1)}$ **16.** $f(x) = \sqrt{x^4 - 16x^2}$

In Exercises 17–20, find the range of the function.

17. $f(x) = 10 - x^2$ **18.** $g(x) = 5 + \sqrt{4 - x}$

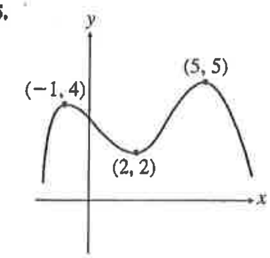
19. $f(x) = \dfrac{x^2}{1 - x^2}$ **20.** $g(x) = \dfrac{3 + x^2}{4 - x^2}$

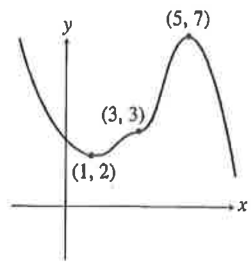
In Exercises 21–24, graph the function and tell whether or not it has a point of discontinuity at $x = 0$. If there is a discontinuity, tell whether it is removable or nonremovable.

21. $g(x) = \dfrac{3}{x}$ **22.** $h(x) = \dfrac{x^3 + x}{x}$

23. $f(x) = \dfrac{|x|}{x}$ **24.** $g(x) = \dfrac{x}{x - 2}$

In Exercises 25–28, state whether each labeled point identifies a local minimum, a local maximum, or neither. Identify intervals on which the function is decreasing and increasing.

25.


26.


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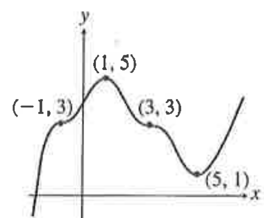
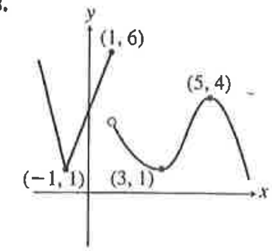


28.


In Exercises 29–34, graph the function and identify intervals on which the function is increasing, decreasing, or constant.

29. $f(x) = |x + 2| - 1$

30. $f(x) = |x + 1| + |x - 1| - 3$

31. $g(x) = |x + 2| + |x - 1| - 2$

32. $h(x) = 0.5(x + 2)^2 - 1$

33. $g(x) = 3 - (x - 1)^2$

34. $f(x) = x^3 - x^2 - 2x$

In Exercises 35–40, determine whether the function is bounded above, bounded below, or bounded on its domain.

35. $y = 32$ **36.** $y = 2 - x^2$

37. $y = 2^x$ **38.** $y = 2^{-x}$

39. $y = \sqrt{1 - x^2}$ **40.** $y = x - x^3$

In Exercises 41–46, use a grapher to find all local maxima and minima and the values of x where they occur. Give values rounded to two decimal places.

41. $f(x) = 4 - x + x^2$ **42.** $g(x) = x^3 - 4x + 1$

43. $h(x) = -x^3 + 2x - 3$ **44.** $f(x) = (x + 3)(x - 1)^2$

45. $h(x) = x^2\sqrt{x + 4}$ **46.** $g(x) = x|2x + 5|$

In Exercises 47–54, state whether the function is odd, even, or neither. Support graphically and confirm algebraically.

47. $f(x) = 2x^4$

48. $g(x) = x^3$

49. $f(x) = \sqrt{x^2 + 2}$

50. $g(x) = \dfrac{3}{1 + x^2}$

51. $f(x) = -x^2 + 0.03x + 5$

52. $f(x) = x^3 + 0.04x^2 + 3$

53. $g(x) = 2x^3 - 3x$

54. $h(x) = \dfrac{1}{x}$

In Exercises 55–62, use a method of your choice to find all horizontal and vertical asymptotes of the function.

55. $f(x) = \dfrac{x}{x - 1}$

56. $q(x) = \dfrac{x - 1}{x}$

57. $g(x) = \dfrac{x + 2}{3 - x}$

58. $q(x) = 1.5^x$

59. $f(x) = \dfrac{x^2 + 2}{x^2 - 1}$

60. $p(x) = \dfrac{4}{x^2 + 1}$

61. $g(x) = \dfrac{4x - 4}{x^3 - 8}$

62. $h(x) = \dfrac{2x - 4}{x^2 - 4}$

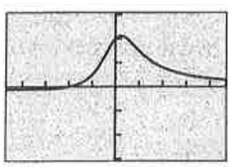
In Exercises 63-66, match the function with the corresponding graph by considering end behavior and asymptotes. All graphs are shown in the same viewing window.

63. $y = \dfrac{x + 2}{2x + 1}$

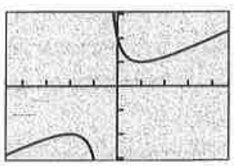
64. $y = \dfrac{x^2 + 2}{2x + 1}$

65. $y = \dfrac{x + 2}{2x^2 + 1}$

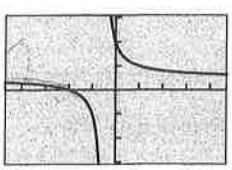
66. $y = \dfrac{x^3 + 2}{2x^2 + 1}$



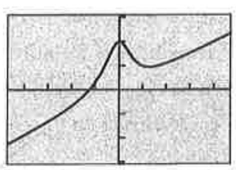
[–4.7, 4.7] by [–3.1, 3.1]
(a)



[–4.7, 4.7] by [–3.1, 3.1]
(c)



[–4.7, 4.7] by [–3.1, 3.1]
(b)



[–4.7, 4.7] by [–3.1, 3.1]
(d)

67. Can a graph cross its own asymptote? The Greek roots of the word "asymptote" mean "not meeting," since graphs tend to approach, but not meet, their asymptotes. Which of the following functions have graphs that *do* intersect their horizontal asymptotes?

(a) $f(x) = \dfrac{x}{x^2 - 1}$

(b) $g(x) = \dfrac{x}{x^2 + 1}$

(c) $h(x) = \dfrac{x^2}{x^3 + 1}$

68. Can a graph have two horizontal asymptotes? Although most graphs have at most one horizontal asymptote, it is possible for a graph to have more than one. Which of the following functions have graphs with more than one horizontal asymptote?

(a) $f(x) = \dfrac{|x^3 + 1|}{8 - x^3}$

(b) $g(x) = \dfrac{|x - 1|}{x^2 - 4}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

69. Can a graph intersect its own vertical asymptote? Graph the function $f(x) = \dfrac{x - |x|}{x^2} + 1$.

(a) The graph of this function does not intersect its vertical asymptote. Explain why it does not.

(b) Show how you can add a single point to the graph of f and get a graph that *does* intersect its vertical asymptote.

(c) Is the graph in (b) the graph of a function?

70. Writing to Learn Explain why a graph cannot have more than two horizontal asymptotes.

Standardized Test Questions

71. True or False The graph of function f is defined as the set of all points $(x, f(x))$ where x is in the domain of f. Justify your answer.

72. True or False A relation that is symmetric with respect to the x-axis cannot be a function. Justify your answer.

In Exercises 73–76, answer the question without using a calculator.

73. Multiple Choice Which function is continuous?

(a) Number of children enrolled in a particular school as a function of time

(b) Outdoor temperature as a function of time

(c) Cost of U.S. postage as a function of the weight of the letter

(d) Price of a stock as a function of time

(e) Number of soft drinks sold at a ballpark as a function of outdoor temperature

74. Multiple Choice Which function is *not* continuous?

(a) Your altitude as a function of time while flying from Reno to Dallas

(b) Time of travel from Miami to Pensacola as a function of driving speed

(c) Number of balls that can fit completely inside a particular box as a function of the radius of the balls

(d) Area of a circle as a function of radius

(e) Weight of a particular baby as a function of time after birth

75. Decreasing Function Which function is decreasing?

(a) Outdoor temperature as a function of time

(b) The Dow Jones Industrial Average as a function of time

(c) Air pressure in the Earth's atmosphere as a function of altitude

(d) World population since 1900 as a function of time

(e) Water pressure in the ocean as a function of depth

76. Increasing or Decreasing Which function cannot be classified as either increasing or decreasing?

(a) Weight of a lead brick as a function of volume

(b) Height of a ball that has been tossed upward as a function of time

(c) Time of travel from Buffalo to Syracuse as a function of driving speed

(d) Area of a square as a function of side length

(e) Height of a swinging pendulum as a function of time

Explorations

77. Bounded Functions As promised in Example 7 of this section, we will give you a chance to prove algebraically that $p(x) = x/(1 + x^2)$ is bounded.

(a) Graph the function and find the smallest integer k that appears to be an upper bound.

(b) Verify that $x/(1 + x^2) < k$ by proving the equivalent inequality $kx^2 - x + k > 0$. (Use the quadratic formula to show that the quadratic has no real zeros.)

(c) From the graph, find the greatest integer k that appears to be a lower bound.

(d) Verify that $x/(1 + x^2) > k$ by proving the equivalent inequality $kx^2 - x + k < 0$.

78. Baylor School Grade Point Averages Baylor School uses a sliding scale to convert the percentage grades on its transcripts to grade point averages (GPAs). Table 1.9 shows the GPA equivalents for selected grades:

Table 1.9 Converting Grades

Grade (x)	GPA (y)
60	0.00
65	1.00
70	2.05
75	2.57
80	3.00
85	3.36
90	3.69
95	4.00
100	4.28

Source: Baylor School College Counselor.

(a) Considering GPA (y) as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?

(b) Make a table showing the *change* (Δy) in GPA as you move down the list. (See Exploration 1.)

(c) Make a table showing the change in Δy as you move down the list. (This is $\Delta\Delta y$.) Considering the *change* (Δy) in GPA as a function of percentage grade (x), is it increasing, decreasing, constant, or none of these?

(d) In general, what can you say about the shape of the graph if y is an increasing function of x and Δy is a decreasing function of x?

(e) Sketch the graph of a function y of x such that y is a decreasing function of x and Δy is an increasing function of x.

79. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is continuous for all x;

(b) f is increasing on $(-\infty, 0]$ and on $[3, 5]$;

(c) f is decreasing on $[0, 3]$ and on $[5, \infty)$;

(d) $f(0) = f(5) = 2$;

(e) $f(3) = 0$.

80. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is decreasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$;

(b) f has a nonremovable point of discontinuity at $x = 0$;

(c) f has a horizontal asymptote at $y = 1$;

(d) $f(0) = 0$;

(e) f has a vertical asymptote at $x = 0$.

81. Group Activity Sketch (freehand) a graph of a function f with domain all real numbers that satisfies all of the following conditions:

(a) f is continuous for all x;

(b) f is an even function;

(c) f is increasing on $[0, 2]$ and decreasing on $[2, \infty)$;

(d) $f(2) = 3$.

82. Group Activity Get together with your classmates in groups of two or three. Sketch a graph of a function, but do not show it to the other members of your group. Using the language of functions (as in Exercises 79–81), describe your function as completely as you can. Exchange descriptions with the others in your group and see if you can reproduce each other's graphs.

Extending the Ideas

83. A function that is bounded above has an infinite number of upper bounds, but there is always a *least upper bound*, i.e., an upper bound that is less than all the others. This least upper bound may or may not be in the range of f. For each of the following functions, find the least upper bound and tell whether or not it is in the range of the function.

(a) $f(x) = 2 - 0.8x^2$

(b) $g(x) = \frac{3x^2}{3 + x^2}$

(c) $h(x) = \frac{1 - x}{x^2}$

(d) $p(x) = 2 \sin (x)$

(e) $q(x) = \frac{4x}{x^2 + 2x + 1}$

84. Writing to Learn A continuous function f has domain all real numbers. If $f(-1) = 5$ and $f(1) = -5$, explain why f must have at least one zero in the interval $[-1, 1]$. (This generalizes to a property of continuous functions known as the Intermediate Value Theorem.)

85. Proving a Theorem Prove that the graph of every odd function with domain all real numbers must pass through the origin.

86. Finding the Range Graph the function $f(x) = \frac{3x^2 - 1}{2x^2 + 1}$ in the window $[-6, 6]$ by $[-2, 2]$.

(a) What is the apparent horizontal asymptote of the graph?

(b) Based on your graph, determine the apparent range of f.

(c) Show algebraically that $-1 \leq \frac{3x^2 - 1}{2x^2 + 1} < 1.5$ for all x, thus confirming your conjecture in part (b).

1.3 TWELVE BASIC FUNCTIONS

What you'll learn about

- What Graphs Can Tell Us
- Twelve Basic Functions
- Analyzing Functions Graphically

. . . and why

As you continue to study mathematics, you will find that the twelve basic functions presented here will come up again and again. By knowing their basic properties, you will recognize them when you see them.

What Graphs Can Tell Us

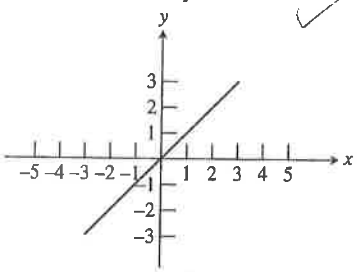
The preceding section has given us a vocabulary for talking about functions and their properties. We have an entire book ahead of us to study these functions in depth, but in this section we want to set the scene by just *looking* at the graphs of twelve "basic" functions that are available on your graphing calculator.

You will find that function attributes such as domain, range, continuity, asymptotes, extrema, increasingness, decreasingness, and end behavior are every bit as graphical as they are algebraic. Moreover, the visual cues are often much easier to spot than the algebraic ones.

In future chapters you will learn more about the algebraic properties that make these functions behave as they do. Only then will you able to *prove* what is visually apparent in these graphs.

Twelve Basic Functions

The Identity Function

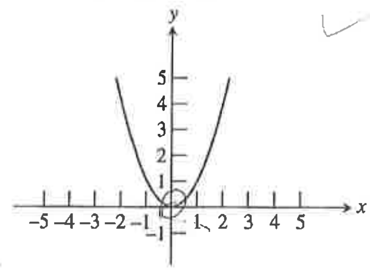


$f(x) = x$

Interesting fact: This is the only function that acts on every real number by leaving it alone.

FIGURE 1.36

The Squaring Function

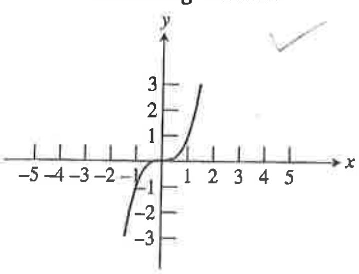


$f(x) = x^2$

Interesting fact: The graph of this function, called a parabola, has a reflection property that is useful in making flashlights and satellite dishes.

FIGURE 1.37

The Cubing Function

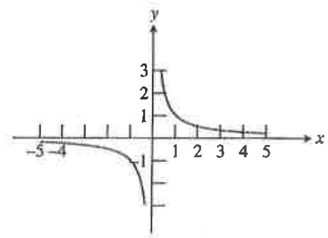


$f(x) = x^3$

Interesting fact: The origin is called a "point of inflection" for this curve because the graph changes curvature at that point.

FIGURE 1.38

The Reciprocal Function

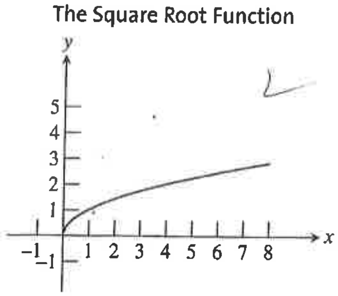


$f(x) = \frac{1}{x}$

Interesting fact: This curve, called a hyperbola, also has a reflection property that is useful in satellite dishes.

FIGURE 1.39

The Square Root Function

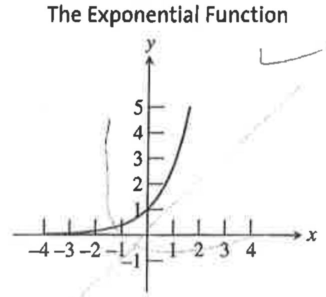


$f(x) = \sqrt{x}$

Interesting fact: Put any positive number into your calculator. Take the square root. Then take the square root again. Then take the square root again, and so on. Eventually you will always get 1.

FIGURE 1.40

The Exponential Function

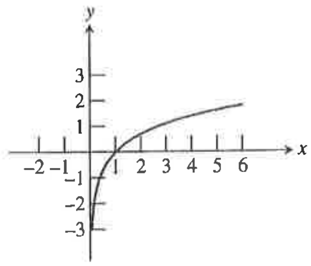


$f(x) = e^x$

Interesting fact: The number e is an irrational number (like π) that shows up in a variety of applications. The symbols e and π were both brought into popular use by the great Swiss mathematician Leonhard Euler (1707–1783).

FIGURE 1.41

The Natural Logarithm Function

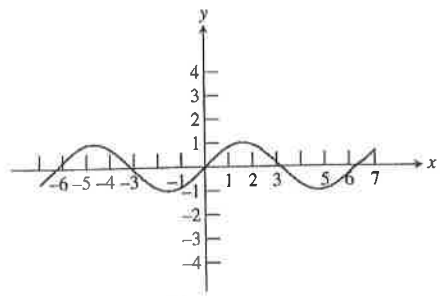


$f(x) = \ln x$

Interesting fact: This function increases very slowly. If the x-axis and y-axis were both scaled with unit lengths of one inch, you would have to travel more than two and a half miles along the curve just to get a foot above the x-axis.

FIGURE 1.42

The Sine Function

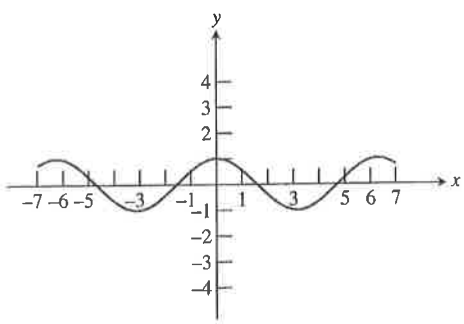


$f(x) = \sin x$

Interesting fact: This function and the sinus cavities in your head derive their names from a common root: the Latin word for "bay." This is due to a 12th-century mistake made by Robert of Chester, who translated a word incorrectly from an Arabic manuscript.

FIGURE 1.43

The Cosine Function

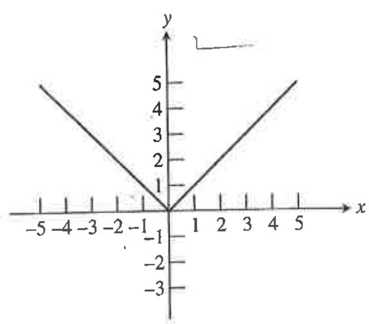


$f(x) = \cos x$

Interesting fact: The local extrema of the cosine function occur exactly at the zeros of the sine function, and vice versa.

FIGURE 1.44

The Absolute Value Function

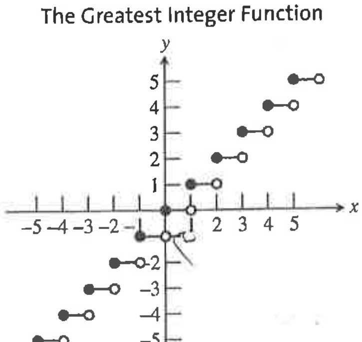


$f(x) = |x| = \text{abs}\,(x)$

Interesting fact: This function has an abrupt change of direction (a "corner") at the origin, while our other functions are all "smooth" on their domains.

FIGURE 1.45

The Greatest Integer Function

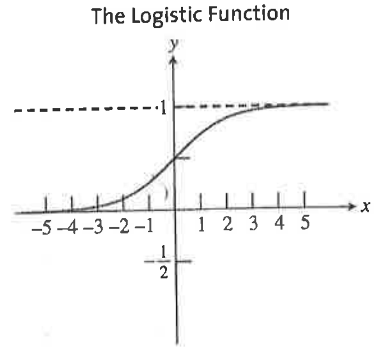


$f(x) = \text{int}\,(x)$

Interesting fact: This function has a jump discontinuity at every integer value of x. Similar-looking functions are called *step functions.*

FIGURE 1.46

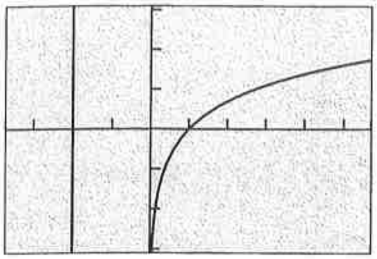
The Logistic Function



$$f(x) = \frac{1}{1 + e^{-x}}$$

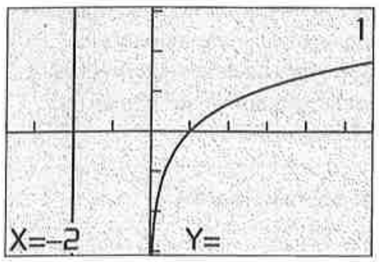
Interesting fact: There are two horizontal asymptotes, the x-axis and the line $y = 1$. This function provides a model for many applications in biology and business.

FIGURE 1.47



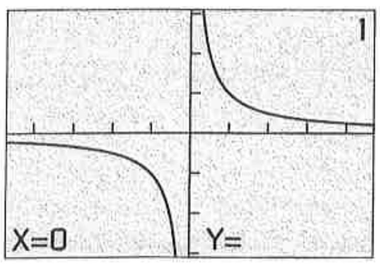
[−3.7, 5.7] by [−3.1, 3.1]

(a)



[−3.7, 5.7] by [−3.1, 3.1]

(b)



[−4.7, 4.7] by [−3.1, 3.1]

(c)

FIGURE 1.48 (a) A vertical line through -2 on the x-axis appears to miss the graph of $y = \ln x$. (b) A TRACE confirms that -2 is not in the domain. (c) A TRACE at $x = 0$ confirms that 0 is not in the domain of $y = 1/x$. (Example 1)

EXAMPLE 1 Looking for domains

(a) Nine of the functions have domain the set of all real numbers. Which three do not?

(b) One of the functions has domain the set of all reals except 0. Which function is it, and why isn't zero in its domain?

(c) Which two functions have no negative numbers in their domains? Of these two, which one is defined at zero?

SOLUTION

(a) Imagine dragging a vertical line along the x-axis. If the function has domain the set of all real numbers, then the line will always intersect the graph. The intersection might occur off screen, but the TRACE function on the calculator will show the y-coordinate if there is one. Looking at the graphs in Figures 1.39, 1.40, and 1.42, we conjecture that there are vertical lines that do not intersect the curve. A TRACE at the suspected x-coordinates confirms our conjecture (Figure 1.48). The functions are $y = 1/x$, $y = \sqrt{x}$ and $y = \ln x$.

(b) The function $y = 1/x$, with a vertical asymptote at $x = 0$, is defined for all real numbers except 0. This is explained algebraically by the fact that division by zero is undefined.

(c) The functions $y = \sqrt{x}$ and $y = \ln x$ have no negative numbers in their domains. (We already knew that about the square root function.) While 0 is in the domain of $y = \sqrt{x}$, we can see by tracing that it is not in the domain of $y = \ln x$. We will see the algebraic reason for this in Chapter 3.

Now try Exercise 13.

EXAMPLE 2 Looking for continuity

Only two of twelve functions have points of discontinuity. Are these points in the domain of the function?

SOLUTION All of the functions have continuous, unbroken graphs except for $y = 1/x$, and $y = \text{int}(x)$.

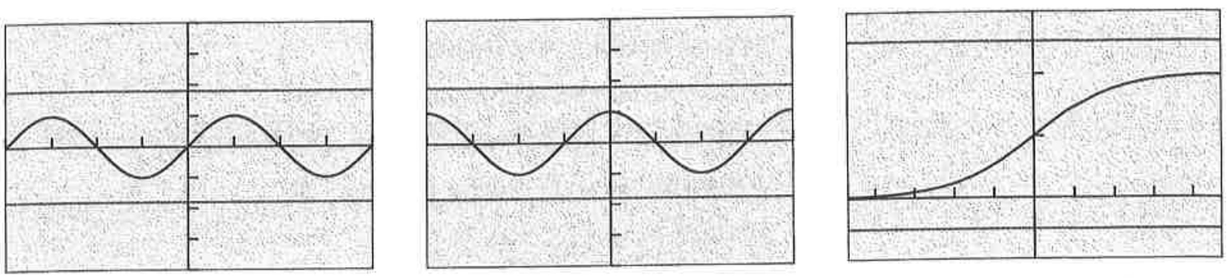
The graph of $y = 1/x$ clearly has an infinite discontinuity at $x = 0$ (Figure 1.39). We saw in Example 1 that 0 is not in the domain of the function. Since $y = 1/x$ is continuous for every point *in its domain*, it is called a **continuous function**.

The graph of $y = \text{int}(x)$ has a discontinuity at every integer value of x (Figure 1.46). Since this function has discontinuities at points in its domain, it is *not* a continuous function. Now try Exercise 15.

EXAMPLE 3 Looking for boundedness

Only three of the twelve basic functions are bounded (above and below). Which three?

SOLUTION A function that is bounded must have a graph that lies entirely between two horizontal lines. The sine, cosine, and logistic functions have this property (Figure 1.49). It looks like the graph of $y = \sqrt{x}$ might also have this property, but we know that the end behavior of the square root function is unbounded: $\lim_{x \to \infty} \sqrt{x} = \infty$, so it is really only bounded below. You will learn in Chapter 4 why the sine and cosine functions are bounded. Now try Exercise 17.



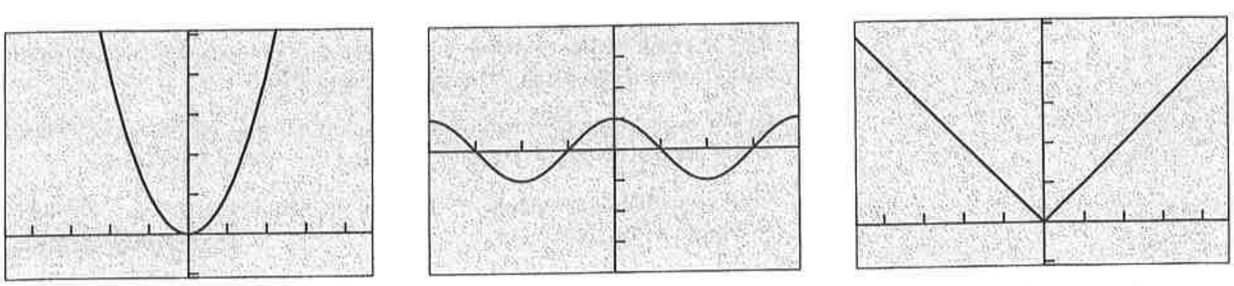
$[-2\pi, 2\pi]$ by $[-4, 4]$ (a) — $[-2\pi, 2\pi]$ by $[-4, 4]$ (b) — $[-4.7, 4.7]$ by $[-0.5, 1.5]$ (c)

FIGURE 1.49 The graphs of $y = \sin x$, $y = \cos x$, and $y = 1/(1 + e^{-x})$ lie entirely between two horizontal lines and are therefore graphs of bounded functions. (Example 3)

EXAMPLE 4 Looking for symmetry

Three of the twelve basic functions are even. Which are they?

SOLUTION Recall that the graph of an even function is symmetric with respect to the y-axis. Three of the functions exhibit the required symmetry: $y = x^2$, $y = \cos x$, and $y = |x|$ (Figure 1.50). Now try Exercise 19.



$[-4.7, 4.7]$ by $[-1.1, 5.1]$ (a) — $[-2\pi, 2\pi]$ by $[-4, 4]$ (b) — $[-4.7, 4.7]$ by $[-1.1, 5.1]$ (c)

FIGURE 1.50 The graphs of $y = x^2$, $y = \cos x$, and $y = |x|$ are symmetric with respect to the y-axis, indicating that the functions are even. (Example 4)

Analyzing Functions Graphically

We could continue to explore the twelve basic functions as in the first four examples, but we also want to make the point that there is no need to restrict ourselves to the basic twelve. We can alter the basic functions slightly and see what happens to their graphs, thereby gaining further visual insights into how functions behave.

EXAMPLE 5 Analyzing a function graphically

Graph the function $y = (x - 2)^2$. Then answer the following questions:

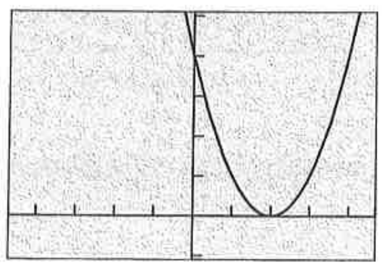
(a) On what interval is the function increasing? On what interval is it decreasing?

(b) Is the function odd, even, or neither?

(c) Does the function have any extrema?

(d) How does the graph relate to the graph of the basic function $y = x^2$?

SOLUTION The graph is shown in Figure 1.51.



[–4.7, 4.7] by [–1.1, 5.1]

FIGURE 1.51 The graph of $y = (x - 2)^2$. (Example 5)

(a) The function is increasing if its graph is headed upward as it moves from left to right. We see that it is increasing on the interval $[2, \infty)$. The function is decreasing if its graph is headed downward as it moves from left to right. We see that it is decreasing on the interval $(-\infty, 2]$.

(b) The graph is not symmetric with respect to the y-axis, nor is it symmetric with respect to the origin. The function is neither.

(c) Yes, we see that the function has a minimum value of 0 at $x = 2$. (This is easily confirmed by the algebraic fact that $(x - 2)^2 \geq 0$ for all x.)

(d) We see that the graph of $y = (x - 2)^2$ is just the graph of $y = x^2$ moved two units to the right. Now try Exercise 35.

EXPLORATION 1 **Looking for Asymptotes**

1. Two of the basic functions have vertical asymptotes at $x = 0$. Which two?
2. Form a new function by adding these functions together. Does the new function have a vertical asymptote at $x = 0$?
3. Three of the basic functions have horizontal asymptotes at $y = 0$. Which three?
4. Form a new function by adding these functions together. Does the new function have a horizontal asymptote at $y = 0$?
5. Graph $f(x) = 1/x$, $g(x) = 1/(2x^2 - x)$, and $h(x) = f(x) + g(x)$. Does $h(x)$ have a vertical asymptote at $x = 0$?

EXAMPLE 6 Identifying a piecewise-defined function

Which of the twelve basic functions has the following **piecewise** definition over separate intervals of its domain?

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

SOLUTION You may recognize this as the definition of the absolute value function (Chapter P). Or, you can reason that the graph of this function must look just like the line $y = x$ to the right of the y-axis, but just like the graph of the line $y = -x$ to the left of the y-axis. That is a perfect description of the absolute value graph in Figure 1.45. Either way, we recognize this as a piecewise definition of $f(x) = |x|$. Now try Exercise 45.

EXAMPLE 7 Defining a function piecewise

Using basic functions from this section, construct a piecewise definition for the function whose graph is shown in Figure 1.52. Is your function continuous?

SOLUTION This appears to be the graph of $y = x^2$ to the left of $x = 0$ and the graph of $y = \sqrt{x}$ to the right of $x = 0$. We can therefore define it piecewise as

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

The function is continuous. Now try Exercise 47.

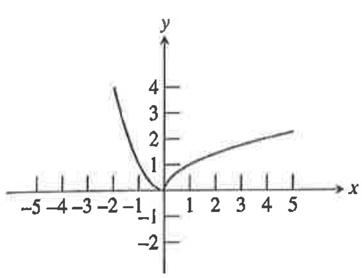
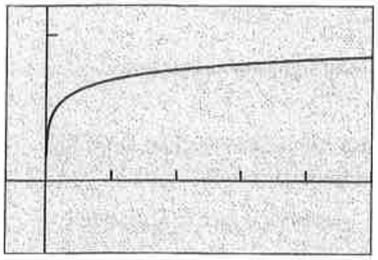


FIGURE 1.52 A piecewise-defined function. (Example 7)

You can go a long way toward understanding a function's behavior by looking at its graph. We will continue that theme in the exercises and then revisit it throughout the book. However, you can't go *all* the way toward understanding a function by looking at its graph, as Example 8 shows.



[–600, 5000] by [–5, 12]

FIGURE 1.53 The graph of $y = \ln x$ still appears to have a horizontal asymptote, despite the much larger window than in Figure 1.43. (Example 8)

EXAMPLE 8 Looking for a horizontal asymptote

Does the graph of $y = \ln x$ (Figure 1.42) have a horizontal asymptote?

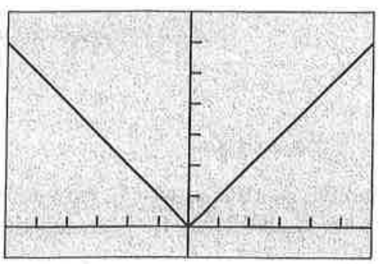
SOLUTION In Figure 1.42 it certainly *looks* like there is a horizontal asymptote that the graph is approaching from below. If we choose a much larger window (Figure 1.53), it still looks that way. In fact, we could zoom out on this function all day long and it would *always* look like it is approaching some horizontal asymptote—but it is not. We will show algebraically in Chapter 3 that the end behavior of this function is $\lim_{x\to\infty} \ln x = \infty$, so its graph must eventually rise above the level of any horizontal line. That rules out any horizontal asymptote, even though there is no *visual* evidence of that fact that we can see by looking at its graph.

Now try Exercise 55.

EXAMPLE 9 Analyzing a function

Give a complete analysis of the basic function $f(x) = |x|$.

SOLUTION



[–6, 6] by [–1, 7]

FIGURE 1.54 The graph of $f(x) = |x|$.

BASIC FUNCTION The Absolute Value Function

$f(x) = |x|$
Domain: All reals
Range: $[0, \infty)$
Continuous
Decreasing on $(\infty, 0]$; increasing on $[0, \infty)$
Symmetric with respect to the y-axis (an even function)
Bounded below
Local minimum at $(0, 0)$
No horizontal asymptotes
No vertical asymptotes
End behavior: $\lim_{x\to\infty} |x| = \infty$ and $\lim_{x\to\infty} |x| = \infty$

Now try Exercise 67.

QUICK REVIEW 1.3 *(For help, go to Sections P.1, P.2, 3.1, and 3.3.)*

In Exercises 1–10, evaluate the expression without using a calculator.

1. $|-59.34|$
2. $|5 - \pi|$
3. $|\pi - 7|$
4. $\sqrt{(-3)^2}$
5. $\ln(1)$
6. e^0
7. $(\sqrt[3]{3})^3$
8. $\sqrt[3]{(-15)^3}$
9. $\sqrt[3]{-8^2}$
10. $|1 - \pi| - \pi$

SECTION 1.3 EXERCISES

In Exercises 1–12, each graph is a slight variation on the graph of one of the twelve basic functions described in this section. Match the graph to one of the twelve functions (a)–(l) and then support your answer by checking the graph on your calculator. (All graphs are shown in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$.)

(a) $y = -\sin x$ **(b)** $y = \cos x + 1$ **(c)** $y = e^x - 2$
(d) $y = (x + 2)^3$ **(e)** $y = x^3 + 1$ **(f)** $y = (x - 1)^2$
(g) $y = |x| - 2$ **(h)** $y = -1/x$ **(i)** $y = -x$
(j) $y = -\sqrt{x}$ **(k)** $y = \text{int}(x + 1)$ **(l)** $y = 2 - 4/(1+e^{-x})$

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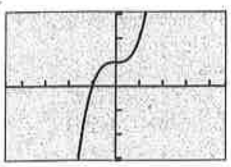

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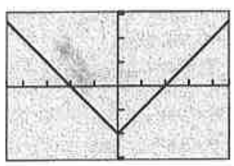

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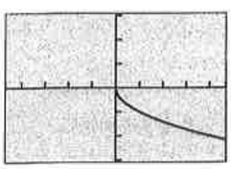

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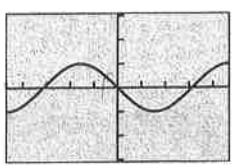

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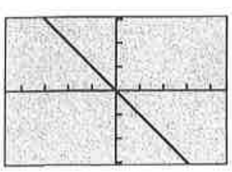

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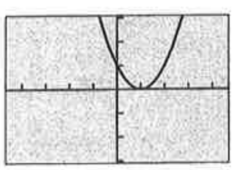

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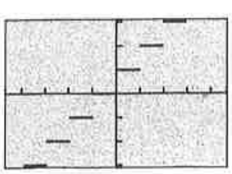

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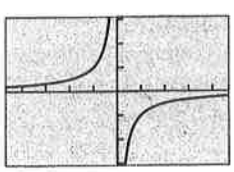

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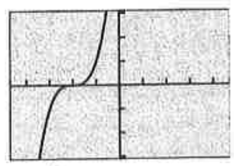

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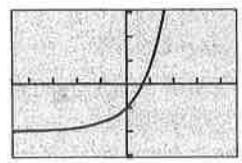

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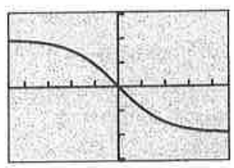

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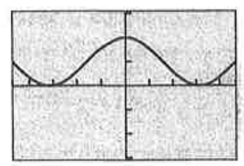


In Exercises 13–18, identify which of Exercises 1–12 display functions that fit the description given.

13. The function whose domain excludes zero.

14. The function whose domain consists of all nonnegative real numbers.

15. The two functions that have at least one point of discontinuity.

16. The function that is not a *continuous function.*

17. The six functions that are bounded below.

18. The four functions that are bounded above.

In Exercises 19–28, identify which of the twelve basic functions fit the description given.

19. The four functions that are odd.

20. The six functions that are increasing on their entire domains.

21. The three functions that are decreasing on the interval $(-\infty, 0)$.

22. The three functions with infinitely many local extrema.

23. The three functions with no zeros.

24. The three functions with range {all real numbers}.

25. The four functions that do *not* have end behavior $\lim_{x\to+\infty} f(x) = +\infty$.

26. The three functions with end behavior $\lim_{x\to-\infty} f(x) = -\infty$.

27. The four functions whose graphs look the same when turned upside-down and flipped about the y-axis.

28. The two functions whose graphs are identical except for a horizontal shift.

In Exercises 29–34, use your graphing calculator to produce a graph of the function. Then determine the domain and range of the function by looking at its graph.

29. $f(x) = x^2 - 5$

30. $g(x) = |x - 4|$

31. $h(x) = \ln(x + 6)$

32. $k(x) = 1/x + 3$

33. $s(x) = \text{int}(x/2)$

34. $p(x) = (x + 3)^2$

In Exercises 35–42, graph the function. Then answer the following questions:

(a) On what interval, if any, is the function increasing? Decreasing?

(b) Is the function odd, even, or neither?

(c) Give the function's extrema, if any.

(d) How does the graph relate to a graph of one of the twelve basic functions?

35. $r(x) = \sqrt{x - 10}$

36. $f(x) = \sin(x) + 5$

37. $f(x) = 3/(1+e^{-x})$

38. $q(x) = e^x + 2$

39. $h(x) = |x| - 10$

40. $g(x) = 4\cos(x)$

41. $s(x) = |x - 2|$

42. $f(x) = 5 - \text{abs}(x)$

43. Find the horizontal asymptotes for the graph shown in Exercise 11.

44. Find the horizontal asymptotes for the graph of $f(x)$ in Exercise 37.

In Exercises 45–52, sketch the graph of the piecewise-defined function. (Try doing it without a calculator.) In each case, give any points of discontinuity.

45. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

46. $g(x) = \begin{cases} x^3 & \text{if } x \le 0 \\ e^x & \text{if } x > 0 \end{cases}$

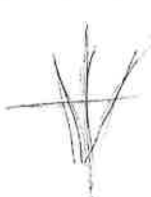
47. $h(x) = \begin{cases} |x| & \text{if } x < 0 \\ \sin x & \text{if } x \ge 0 \end{cases}$

48. $w(x) = \begin{cases} 1/x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \ge 0 \end{cases}$

49. $f(x) = \begin{cases} \cos x & \text{if } x \le 0 \\ e^x & \text{if } x > 0 \end{cases}$

50. $g(x) = \begin{cases} |x| & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$

51. $f(x) = \begin{cases} -3 - x & \text{if } x \le 0 \\ 1 & \text{if } 0 < x < 1 \\ x^2 & \text{if } x \ge 1 \end{cases}$



52. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ |x| & \text{if } -1 \le x < 1 \\ \text{int}(x) & \text{if } x \ge 1 \end{cases}$

53. Writing to Learn The function $f(x) = \sqrt{x^2}$ is one of our twelve basic functions written in another form.

(a) Graph the function and identify which basic function it is.

(b) Explain algebraically why the two functions are equal.

54. Uncovering Hidden Behavior The function $g(x) = \sqrt{x^2 + 0.0001} - 0.01$ is *not* one of our twelve basic functions written in another form.

(a) Graph the function and identify which basic function it appears to be.

(b) Verify numerically that it is not the basic function that it appears to be.

55. Writing to Learn The function $f(x) = \ln(e^x)$ is one of our twelve basic functions written in another form.

(a) Graph the function and identify which basic function it is.

(b) Explain how the equivalence of the two functions in (a) shows that the natural logarithm function is *not* bounded above (even though it *appears* to be bounded above in Figure 1.42).

56. Writing to Learn Let $f(x)$ be the function that gives the cost, in cents, to mail a letter that weighs x ounces. As of June 2002, the cost is 37 cents for a letter that weighs up to one ounce, plus 23 cents for each additional ounce of portion thereof.

(a) Sketch a graph of $f(x)$.

(b) How is this function similar to the greatest integer function? How is it different?

57. Analyzing a Function Set your calculator to DOT mode and graph the greatest integer function, $y = \text{int}(x)$, in the window $[-4.7, 4.7]$ by $[-3.1, 3.1]$. Then complete the following analysis.

BASIC FUNCTION The Greatest Integer Function

$f(x) = \text{int}\, x$

Domain:
Range:
Continuity:
Increasing/decreasing behavior:
Symmetry:
Boundedness:
Local extrema:
Horizontal asymptotes:
Vertical asymptotes:
End behavior:

Standardized Test Questions

58. True or False The greatest integer function has an inverse function. Justify your answer.

59. True or False The logistic function has two horizontal asymptotes. Justify your answer.

In Exercises 60–63, you may use a graphing calculator to answer the question.

60. Multiple Choice Which function has range {all real numbers}?

(a) $f(x) = 4 + \ln x$

(b) $f(x) = 3 - 1/x$

(c) $f(x) = 5/(1 + e^{-x})$

(d) $f(x) = \text{int}(x - 2)$

(e) $f(x) = 4 \cos x$

61. Multiple Choice Which function is bounded both above and below?

(a) $f(x) = x^2 - 4$

(b) $f(x) = (x - 3)^3$

(c) $f(x) = 3e^x$

(d) $f(x) = 3 + 1/(1 + e^{-x})$

(e) $f(x) = 4 - |x|$

62. Multiple Choice Which of the following is the same as the restricted-domain function $f(x) = \text{int}(x)$, $0 \le x < 2$?

(a) $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$

(b) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \le 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \end{cases}$

(d) $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x < 2 \end{cases}$

(e) $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 + x & \text{if } 1 \le x < 2 \end{cases}$

63. Multiple Choice Increasing Functions Which function is increasing on the interval $(-\infty, \infty)$?

(a) $f(x) = \sqrt{3 + x}$

(b) $f(x) = \text{int}(x)$

(c) $f(x) = 2x^2$

(d) $f(x) = \sin x$

(e) $f(x) = 3/(1 + e^{-x})$

Explorations

64. Which is Bigger? For positive values of x, we wish to compare the values of the basic functions x^2, x, and $\sqrt{x}$.

(a) How would you order them from least to greatest?

(b) Graph the three functions in the viewing window $[0, 30]$ by $[0, 20]$. Does the graph confirm your response in (a)?

(c) Now graph the three functions in the viewing window $[0, 2]$ by $[0, 1.5]$.

(d) Write a careful response to the question in (a) that accounts for all positive values of x.

65. Odds and Evens There are four odd functions and three even functions in the gallery of twelve basic functions. After multiplying these functions together pairwise in different combinations and exploring the graphs of the products, make a conjecture about the symmetry of:

(a) a product of two odd functions.

(b) a product of two even functions.

(c) a product of an odd function and an even function.

66. Group Activity Assign to each student in the class the name of one of the twelve basic functions, but secretly so that no student knows the "name" of another. (The same function name could be given to several students, but all the functions should be used at least once.) Let each student make a one-sentence self-introduction to the class that reveals something personal "about who I am that really identifies me." The rest of the students then write down their guess as to the function's identity. Hints should be subtle and cleverly anthropomorphic. (For example, the absolute value function saying "I have a very sharp smile" is subtle and clever, while "I am absolutely valuable" is not very subtle at all.)

67. Pepperoni Pizzas For a statistics project, a student counted the number of pepperoni slices on pizzas of various sizes at a local pizzeria, compiling the following table:

TABLE 1.10

Type of Pizza	Radius	Pepperoni count
Personal	4"	12
Medium	6"	27
Large	7"	37
Extra Large	8"	48

(a) Explain why the pepperoni count (P) ought to be proportional to the square of the radius (r).

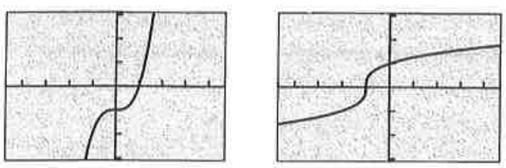
(b) Assuming that $P = k \cdot r^2$, use the data pair (4, 12) to find the value of k.

(c) Does the algebraic model fit the rest of the data well?

(d) Some pizza places have charts showing their kitchen staff how much of each topping should be put on each size of pizza. Do you think this pizzeria uses such a chart? Explain.

Extending the Ideas

68. Inverse Functions Two functions are said to be *inverses* of each other if the graph of one can be obtained from the graph of the other by reflecting it across the line $y = x$. For example, the functions with the graphs shown below are inverses of each other:



[–4.7, 4.7] by [–3.1, 3.1]
(a)

[–4.7, 4.7] by [–3.1, 3.1]
(b)

(a) Two of the twelve basic functions in this section are inverses of each other. Which are they?

(b) Two of the twelve basic functions in this section are their own inverses. Which are they?

(c) If you restrict the domain of one of the twelve basic functions to $[0, \infty)$, it becomes the inverse of another one. Which are they?

69. Identifying a Function by Its Properties

(a) Seven of the twelve basic functions have the property that $f(0) = 0$. Which five do not?

(b) Only one of the twelve basic functions has the property that $f(x + y) = f(x) + f(y)$ for all x and y in its domain. Which one is it?

(c) One of the twelve basic functions has the property that $f(x + y) = f(x)f(y)$ for all x and y in its domain. Which one is it?

(d) One of the twelve basic functions has the property that $f(xy) = f(x) + f(y)$ for all x and y in its domain. Which one is it?

(e) Four of the twelve basic functions have the property that $f(x) + f(-x) = 0$ for all x in their domains. Which four are they?