1.4 BUILDING FUNCTIONS FROM FUNCTIONS

What you'll learn about

- Combining Functions Algebraically
- Composition of Functions
- Relations and Implicity Defined Functions
- Relations Defined Parametrically
- Inverse Relations and Inverse Functions

. . . and why

Most of the functions that you will encounter in calculus and in real life can be created by combining or modifying other functions.

Combining Functions Algebraically

Knowing how a function is "put together" is an important first step when applying the tools of calculus. Functions have their own algebra based on the same operations we apply to real numbers (addition, subtraction, multiplication, and division). One way to build new functions is to apply these operations, using the following definitions.

Definition Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with intersecting domains. Then for all values of x in the intersection, the algebraic combinations of f and g are defined by the following rules:

Sum: $(f + g)(x) = f(x) + g(x)$

Difference: $(f - g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x)g(x)$

Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, provided $g(x) \neq 0$

In each case, the domain of the new function consists of all numbers that belong to both the domain of f and the domain of g. As noted, the zeros of the denominator are excluded from the domain of the quotient.

Euler's function notation works so well in the above definitions that it almost obscures what is really going on. The "+" in the expression "$(f + g)(x)$" stands for a brand new operation called *function addition*. It builds a new function, $f + g$, from the given functions f and g. Like any function, $f + g$ is defined by what it does: It takes a domain value x and returns a range value $f(x) + g(x)$. Note that the "+" sign in "$f(x) + g(x)$" *does* stand for the familiar operation of real number addition. So, with the same symbol taking on different roles on either side of the equal sign, there is more to the above definitions than first meets the eye.

Fortunately, the definitions are easy to apply.

EXAMPLE 1 Defining new functions algebraically

Let $f(x) = x^2$ and $g(x) = \sqrt{x + 1}$.

Find formulas for the functions $f + g$, $f - g$, fg, f/g, and gg. Give the domain of each.

SOLUTION We first determine that f has domain all real numbers and that g has domain $[-1, \infty)$. These domains overlap, the intersection being the interval $[-1, \infty)$. So:

$(f+g)(x) = f(x) + g(x) = x^2 + \sqrt{x+1}$ with domain $[-1, \infty)$.

$(f-g)(x) = f(x) - g(x) = x^2 - \sqrt{x+1}$ with domain $[-1, \infty)$.

$(fg)(x) = f(x)g(x) = x^2\sqrt{x+1}$ with domain $[-1, \infty)$.

$\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^2}{\sqrt{x+1}}$ with domain $(-1, \infty)$.

$(gg)(x) = g(x)g(x) = (\sqrt{x+1})^2$ with domain $[-1, \infty)$.

Note that we could express $(gg)(x)$ more simply as $x + 1$. That would be fine, but the simplification would not change the fact that the domain of gg is (by definition) the interval $[-1, \infty)$. Under other circumstances the function $h(x) = x + 1$ would have domain all real numbers, but under these circumstances it cannot; it is a product of two functions with restricted domains.

Now try Exercise 3.

Composition of Functions

It is not hard to see that the function $\sin(x^2)$ is built from the basic functions $\sin x$ and x^2, but the functions are not put together by addition, subtraction, multiplication, or division. Instead, the two functions are combined by simply applying them in order—first the squaring function, then the sine function. This operation for combining functions, which has no counterpart in the algebra of real numbers, is called *function composition*.

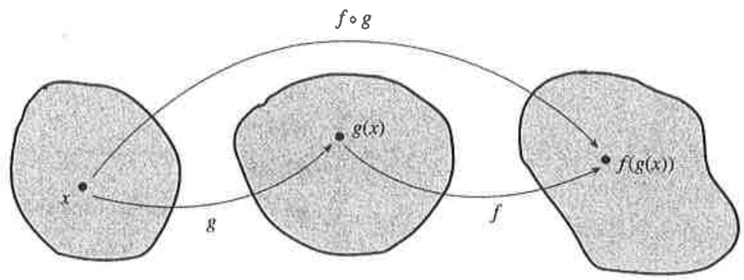
Definition Composition of Functions

Let f and g be two functions such that the domain of f intersects the range of g. The **composition f of g**, denoted $f \circ g$, is defined by the rule

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ consists of all x-values in the domain of g that map to $g(x)$-values in the domain of f. (See Figure 1.55.)

The composition g of f, denoted $g \circ f$, is defined similarly. In most cases $g \circ f$ and $f \circ g$ are different functions. (In the language of algebra, "function composition is not commutative.")



x must be in the domain of g and $g(x)$ must be in the domain of f

FIGURE 1.55 In the composition $f \circ g$, the function g is applied first and then f. This is the reverse of the order in which we read the symbols.

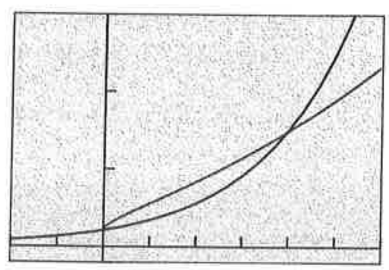
EXAMPLE 2 Composing functions

Let $f(x) = e^x$ and $g(x) = \sqrt{x}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$ and verify that the functions $f \circ g$ and $g \circ f$ are not the same.

SOLUTION

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = e^{\sqrt{x}}$$
$$(g \circ f)(x) = g(f(x)) = g(e^x) = \sqrt{e^x}$$

One verification that these functions are not the same is that they have different domains: $f \circ g$ is defined only for $x \geq 0$, while $g \circ f$ is defined for all real numbers. We could also consider their graphs (Figure 1.56), which agree only at $x = 0$ and $x = 4$.



[–2, 6] by [–1, 15]

FIGURE 1.56 The graphs of $y = e^{\sqrt{x}}$ and $y = \sqrt{e^x}$ are not the same. (Example 2)

Finally, the graphs suggest a numerical verification: Find a single value of x for which $f(g(x))$ and $g(f(x))$ give different values. For example, $f(g(1)) = e$ and $g(f(1)) = \sqrt{e}$. The graph helps us to make a judicious choice of x. You do not want to check the functions at $x = 0$ and $x = 4$ and conclude that they are the same! Now try Exercise 11.

EXAMPLE 3 Finding the domain of a composition

Let $f(x) = x^2 - 1$ and let $g(x) = \sqrt{x}$. Find the domains of the composite functions

(a) $g \circ f$ **(b)** $f \circ g$

SOLUTION

(a) We compose the functions in the order specified:

$$(g \circ f)(x) = g(f(x))$$
$$= \sqrt{x^2 - 1}$$

For x to be in the domain of $g \circ f$, we must first find $f(x) = x^2 - 1$, which we can do for all real x. Then we must take the square root of the result, which we can only do for nonnegative values of $x^2 - 1$.

Therefore, the domain of $g \circ f$ consists of all real numbers for which $x^2 - 1 \geq 0$, namely the union $(-\infty, -1] \cup [1, \infty)$.

(b) Again, we compose the functions in the order specified:

$$(f \circ g)(x) = f(g(x))$$
$$= (\sqrt{x})^2 - 1$$

For x to be in the domain of $f \circ g$, we must first be able to find $g(x) = \sqrt{x}$, which we can only do for nonnegative values of x. Then we must be able to square the result and subtract 1, which we can do for all real numbers.

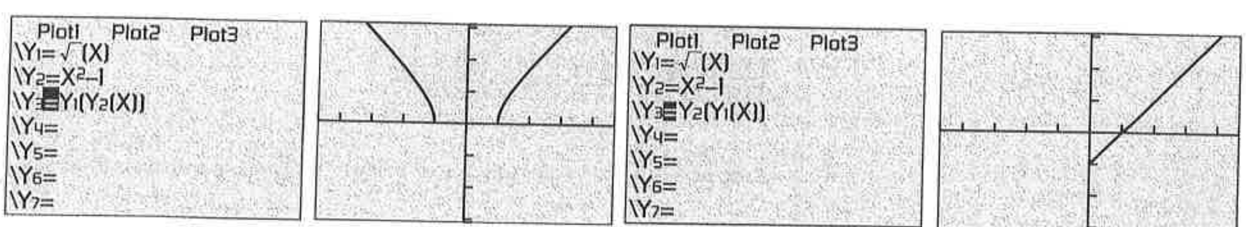
Therefore, the domain of $f \circ g$ consists of the interval $[0, \infty)$.

Support Graphically

We can graph the composition functions to see if the grapher respects the domain restrictions. The screen to the left of each graph shows the set up in the "Y=" editor. Figure 1.57b shows the graph of $y = (g \circ f)(x)$, while Figure 1.57d shows the graph of $y = (f \circ g)(x)$. The graphs support our algebraic work quite nicely. Now try Exercise 13.

CAUTION

We might choose to express $(f \circ g)$ more simply as $x - 1$. However, you must remember that the composition is restricted to the domain of $g(x) = \sqrt{x}$, or $[0, \infty)$. The domain of $x - 1$ is all real numbers. It is a good idea to work out the domain of a composition before you simplify the expression for $f(g(x))$. One way to simplify and maintain the restriction on the domain in Example 3 is to write $(f \circ g)(x) = x - 1, x \geq 0$.



(a) [–4.7, 4.7] by [–3.1, 3.1] (b) (c) [–4.7, 4.7] by [–3.1, 3.1] (d)

FIGURE 1.57 The functions Y1 and Y2 are composed to get the graphs of $y = (g \circ f)(x)$ and $y = (f \circ g)(x)$, respectively. The graphs support our conclusions about the domains of the two composite functions. (Example 3)

In Examples 2 and 3 two functions were *composed* to form new functions. There are times in calculus when we need to reverse the process. That is, we may begin with a function h and *decompose* it by finding functions whose composition is h.

EXAMPLE 4 Decomposing functions

For each function h, find functions f and g such that $h(x) = f(g(x))$.

(a) $h(x) = (x + 1)^2 - 3(x + 1) + 4$

(b) $h(x) = \sqrt{x^3 + 1}$

SOLUTION

(a) We can see that h is quadratic in $x + 1$. Let $f(x) = x^2 - 3x + 4$ and let $g(x) = x + 1$. Then

$$h(x) = f(g(x)) = f(x + 1) = (x + 1)^2 - 3(x + 1) + 4.$$

(b) We can see that h is the square root of the function $x^3 + 1$. Let $f(x) = \sqrt{x}$ and let $g(x) = x^3 + 1$. Then

$$h(x) = f(g(x)) = f(x^3 + 1) = \sqrt{x^3 + 1}.$$

Now try Exercise 17.

There is often more than one way to decompose a function. For example, an alternate way to decompose $h(x) = \sqrt{x^3 + 1}$ in Example 4b is to let $f(x) = \sqrt{x + 1}$ and let $g(x) = x^3$. Then $h(x) = f(g(x)) = f(x^3) = \sqrt{x^3 + 1}$.

FIGURE 1.58 (Example 5)

EXAMPLE 5 Modeling with function composition

In the medical procedure known as angioplasty, doctors insert a catheter into a heart vein (through a large peripheral vein) and inflate a small, spherical balloon on the tip of the catheter. Suppose the balloon is inflated at a constant rate of 44 cubic millimeters per second. (See Figure 1.58.)

(a) Find the volume after t seconds

(b) When the volume is V, what is the radius r?

(c) Write an equation that gives the radius r as a function of the time. What is the radius after 5 seconds?

SOLUTION

(a) After t seconds, the volume will be $44t$.

(b) Solve Algebraically

$$\frac{4}{3}\pi r^3 = v$$

$$r^3 = \frac{3v}{4\pi}$$

$$r = \sqrt[3]{\frac{3v}{4\pi}}$$

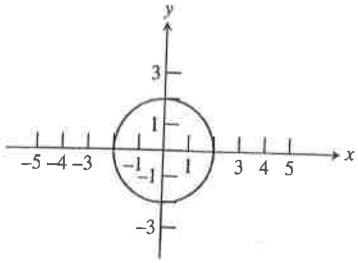


FIGURE 1.59 A circle of radius 2 centered at the origin. This set of ordered pairs (x, y) defines a *relation* that is not a function, because the graph fails the vertical line test.

(c) Substituting $44t$ for V gives $r = \sqrt[3]{\frac{3 \cdot 44t}{4\pi}}$ or $r = \sqrt[3]{\frac{33t}{\pi}}$. After 5 seconds, the radius will be $r = \sqrt[3]{\frac{33 \cdot 5}{\pi}} \approx 3.74$ mm.

Now try Exercise 21.

Relations and Implicitly Defined Functions

There are many useful curves in mathematics that fail the vertical line test and therefore are not graphs of functions. One such curve is the circle in Figure 1.59. While y is not related to x as a function in this instance, there is certainly some sort of relationship going on. In fact, not only does the shape of the graph show a significant *geometric* relationship among the points, but the ordered pairs (x, y) exhibit a significant *algebraic* relationship as well: They consist exactly of the solutions to the equation $x^2 + y^2 = 4$.

The general term for a set of ordered pairs (x, y) is a **relation**. If the relation happens to relate a *single* value of y to each value of x, then the relation is also a function and its graph will pass the vertical line test. In the case of the circle with equation $x^2 + y^2 = 4$, both $(0, 2)$ and $(0, -2)$ are in the relation, so y is not a function of x.

EXAMPLE 6 Verifying pairs in a relation

Determine which of the ordered pairs $(2, -5)$, $(1, 3)$, and $(2, 1)$ are in the relation defined by $x^2y + y^2 = 5$. Is the relation a function?

SOLUTION We simply substitute the x- and y-coordinates of the ordered pairs into $x^2y + y^2$ and see if we get 5.

$(2, -5)$:	$(2)^2(-5) + (-5)^2 = 5$	Substitute $x = 2, y = -5$.
$(1, 3)$:	$(1)^2(3) + (3)^2 = 12 \neq 5$	Substitute $x = 1, y = 3$.
$(2, 1)$:	$(2)^2(1) + (1)^2 = 5$	Substitute $x = 2, y = 1$.

So, $(2, -5)$ and $(2, 1)$ are in the relation, but $(1, 3)$ is not.

Since the equation relates two different y-values (-5 and 1) to the same x-value (2), the relation cannot be a function. Now try Exercise 25.

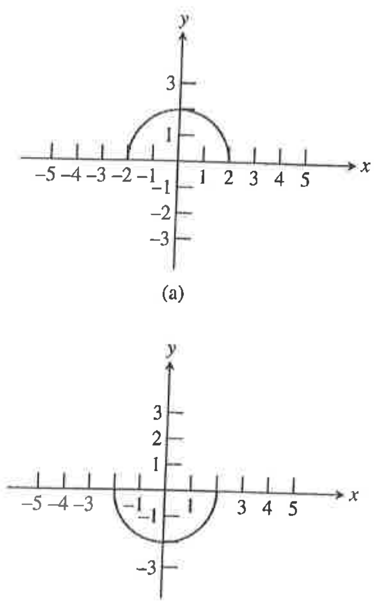


FIGURE 1.60 The graphs of (a) $y = +\sqrt{4 - x^2}$ and (b) $y = -\sqrt{4 - x^2}$. In each case, y is defined as a function of x. These two functions are defined *implicitly* by the relation $x^2 + y^2 = 4$.

Let us revisit the circle $x^2 + y^2 = 4$. While it is not a function itself, we can split it into two equations that *do* define functions, as follows:

$$x^2 + y^2 = 4$$
$$y^2 = 4 - x^2$$
$$y = +\sqrt{4 - x^2} \text{ or } y = -\sqrt{4 - x^2}$$

The graphs of these two functions are, respectively, the upper and lower semicircles of the circle in Figure 1.59. They are shown in Figure 1.60. Since all the ordered pairs in either of these functions satisfy the equation $x^2 + y^2 = 4$, we say that the relation given by the equation defines the two functions **implicitly**.

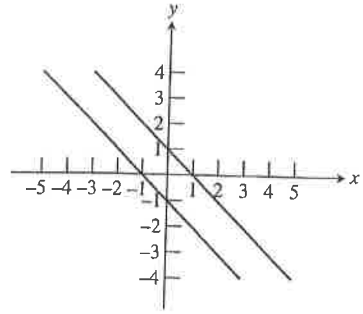


FIGURE 1.61 The graph of the relation $x^2 + 2xy + y^2 = 1$ (Example 7)

EXAMPLE 7 Using implicitly defined functions

Describe the graph of the relation $x^2 + 2xy + y^2 = 1$.

SOLUTION This looks like a difficult task at first, but notice that the expression on the left of the equal sign is a factorable trinomial. This enables us to split the relation into two implicitly defined functions as follows:

$$x^2 + 2xy + y^2 = 1$$

$$(x + y)^2 = 1 \qquad \text{Factor.}$$

$$x + y = \pm 1 \qquad \text{Extract square roots.}$$

$$x + y = 1 \text{ or } x + y = -1$$

$$y = -x + 1 \text{ or } y = -x - 1 \qquad \text{Solve for } y.$$

The graph consists of two parallel lines (Figure 1.61), each the graph of one of the implicitly defined functions. Now try Exercise 27.

Relations Defined Parametrically

Another natural way to define functions or, more generally, relations, is to define *both* elements of the ordered pair (x, y) in terms of another variable t, called a **parameter**. We illustrate with an example.

EXAMPLE 8 Defining a function parametrically

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t + 1$$

$$y = t^2 + 2t$$

where t is any real number.

(a) Find the points determined by $t = -3, -2, -1, 0, 1, 2,$ and 3.

(b) Find an algebraic relationship between x and y. Is y a function of x?

(c) Graph the relation in the (x, y) plane.

SOLUTION

(a) Substitute each value of t into the formulas for x and y to find the point that it determines parametrically:

t	$x = t + 1$	$y = t^2 + 2t$	(x, y)
-3	-2	3	$(-2, 3)$
-2	-1	0	$(-1, 0)$
-1	0	-1	$(0, -1)$
0	1	0	$(1, 0)$
1	2	3	$(2, 3)$
2	3	8	$(3, 8)$
3	4	15	$(4, 15)$

GRAPHING RELATIONS

Relations that are not functions are often not easy to graph. We will study some special cases later in the course (circles, ellipses, etc.), but some simple-looking relations like Example 6 are difficult to graph (see Exercise 78). Nor do our calculators help much, because the equation can not be put into "Y1=" form. Interestingly, we *do* know that the graph of the relation in Example 6, whatever it looks like, fails the vertical line test.

(b) We can find the relationship between x and y algebraically by the method of substitution. First solve for t in terms of x to obtain $t = x - 1$.

$$
\begin{aligned}
y &= t^2 + 2t && \text{Given} \\
y &= (x-1)^2 + 2(x-1) && t = x - 1 \\
&= x^2 - 2x + 1 + 2x - 2 && \text{Expand.} \\
&= x^2 - 1 && \text{Simplify.}
\end{aligned}
$$

This is consistent with the ordered pairs we had found in the table. As t varies over all real numbers, we will get all the ordered pairs in the relation $y = x^2 - 1$, which does indeed define y as a function of x.

(c) Since the parametrically defined relation consists of all ordered pairs in the relation $y = x^2 - 1$, we can get the graph by simply graphing the parabola $y = x^2 - 1$. See Figure 1.62. Now try Exercise 35.

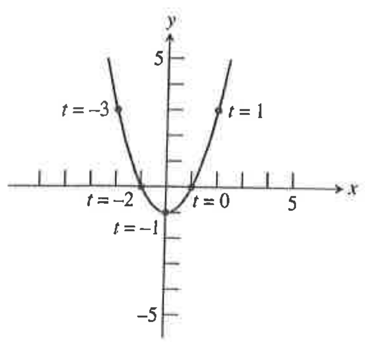


FIGURE 1.62 (Example 8)

EXAMPLE 9 Using a graphing calculator in parametric mode

Consider the set of all ordered pairs (x,y) defined by the equations

$$x = t^2 + 2t$$
$$y = t + 1$$

where t is any real number.

(a) Use a graphing calculator to find the points determined by $t = -3, 2, -1, 0, 1, 2,$ and 3.

(b) Use a graphing calculator to graph the relation in the (x,y) plane.

(c) Is y a function of x?

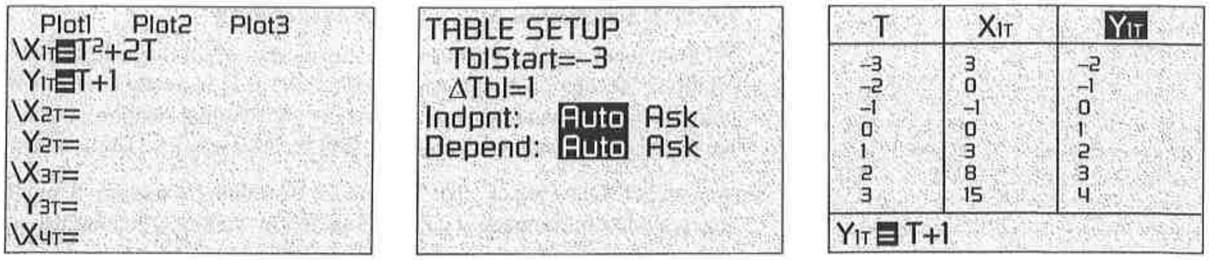
(d) Find an algebraic relationship between x and y.

SOLUTION

(a) When the calculator is in *parametric mode*, the "Y =" screen provides a space to enter both X and Y as functions of the parameter T (Figure 1.63a). After entering the functions, use the table setup in Figure 1.63b to obtain the table shown in Figure 1.63c. The table shows, for example, that when T = −3 we have X1T = 3 and Y1T = −2, so the ordered pair corresponding to $t = -3$ is $(3, -2)$.

t	(x, y)
−3	(3, −2)
−2	(0, −1)
−1	(−1, 0)
0	(0, 1)
1	(3, 2)
2	(8, 3)
3	(15, 4)

(b) In parametric mode, the "WINDOW" screen contains the usual x-axis information, as well as "Tmin," "Tmax," and "Tstep" (Figure 1.64a). To include most of the points listed in part (a), we set Xmin = −5, Xmax = 5, Ymin = −3, and Ymax = 3. Since $t = y - 1$, we set Tmin and Tmax to values one less than those for Ymin and Ymax.



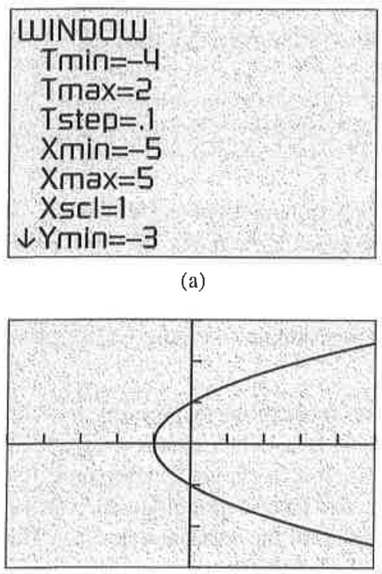
(a) (b)

FIGURE 1.63 Using the table feature of a grapher set in parametric mode. (Example 9)

The value of Tstep determines how far the grapher will go from one value of t to the next as it computes the ordered pairs. With Tmax − Tmin = 6 and Tstep = 0.1, the grapher will compute 60 points, which is sufficient. (The more points, the smoother the graph. See Exploration 1.) The graph is shown in Figure 1.64b. Use trace to find some of the points found in (a).

(c) No, y is not a function of x. We can see this from part (a) because $(0, -1)$ and $(0, 1)$ have the same x-value but different y-values. Alternatively, notice that the graph in (b) fails the vertical line test.

(d) We can use the same algebraic steps as in Example 8 to get the relation in terms of x and y: $x = y^2 - 1$. Now try Exercise 37.



(a)

[−5, 5] by [−3, 3]

(b)

FIGURE 1.64 The graph of a parabola in parametric mode on a graphing calculator. (Example 9)

TIME FOR T

Functions defined by parametric equations are frequently encountered in problems of motion, where the x- and y-coordinates of a moving object are computed as functions of time. This makes time the parameter, which is why you almost always see parameters given as "t" in parametric equations.

EXPLORATION 1 Watching your Tstep

1. Graph the parabola in Example 9 in parametric mode as described in the solution. Press TRACE and observe the values of T, X, and Y. At what value of T does the calculator begin tracing? What point on the parabola results? (It's off the screen.) At what value of T does it stop tracing? What point on the parabola results? How many points are computed as you trace from start to finish?
2. Leave everything else the same and change the Tstep to 0.01. Do you get a smoother graph? Why or why not?
3. Leave everything else the same and change the Tstep to 1. Do you get a smoother graph? Why or why not?
4. What effect does the Tstep have on the speed of the grapher? Is this easily explained?
5. Now change the Tstep to 2. Why does the left portion of the parabola disappear? (It may help to TRACE along the curve.)
6. Change the Tstep back to 0.1 and change the Tmin to −1. Why does the bottom side of the parabola disappear? (Again, it may help to TRACE.)
7. Make a change to the window that will cause the grapher to show the bottom side of the parabola but not the top.

Inverse Relations and Inverse Functions

What happens when we reverse the coordinates of all the ordered pairs in a relation? We obviously get another relation, as it is another set of ordered pairs, but does it bear any resemblance to the original relation? If the original relation happens to be a function, will the new relation also be a function?

We can get some idea of what happens by examining Examples 8 and 9. The ordered pairs in Example 9 can be obtained by simply reversing the coordinates of the ordered pairs in Example 8. This is because we set up Example 9 by switching the parametric equations for x and y that we used in Example 8. We say that the relation in Example 9 is the *inverse relation* of the relation in Example 8.

> **Definition Inverse Relation**
>
> The ordered pair (a, b) is in a relation if and only if the ordered pair (b, a) is in the **inverse relation**.

We will study the connection between a relation and its inverse. We will be most interested in inverse relations that happen to be *functions*. Notice that the graph of the inverse relation in Example 9 fails the vertical line test and is therefore not the graph of a function. Can we predict this failure by considering the graph of the original relation in Example 8? Figure 1.65 suggests that we can.

The inverse graph in Figure 1.65b fails the vertical line test because two different y-values have been paired with the same x-value. This is a direct consequence of the fact that the original relation in Figure 1.65a paired two different x-values with the same y-value. The inverse graph fails the *vertical* line test precisely because the original graph fails the *horizontal* line test. This gives us a test for relations whose inverses are functions.

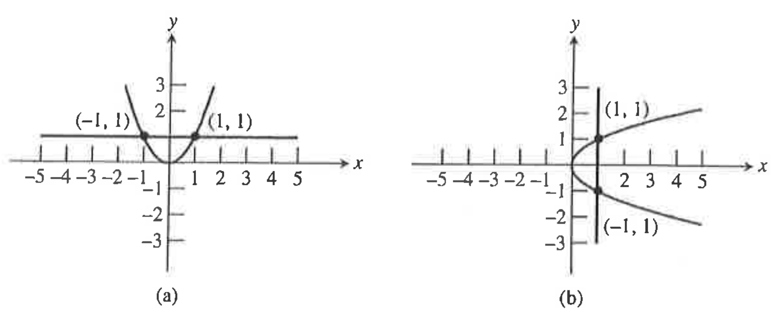


FIGURE 1.65 The inverse relation in (b) fails the vertical line test because the original relation in (a) fails the horizontal line test.

Horizontal Line Test

The inverse of a relation is a function if and only if each horizontal line intersects the graph of the original relation in at most one point.

EXAMPLE 10 Applying the horizontal line test

Which of the graphs (1)–(4) in Figure 1.66 are graphs of

(a) relations that are functions?

(b) relations that have inverses that are functions?

SOLUTION

(a) Graphs (1) and (4) are graphs of functions because these graphs pass the vertical line test. Graphs (2) and (3) are not graphs of functions because these graphs fail the vertical line test.

(b) Graphs (1) and (2) are graphs of relations whose inverses are functions because these graphs pass the horizontal line test. Graphs (3) and (4) fail the horizontal line test so their inverse relations are not functions.

Now try Exercise 39.

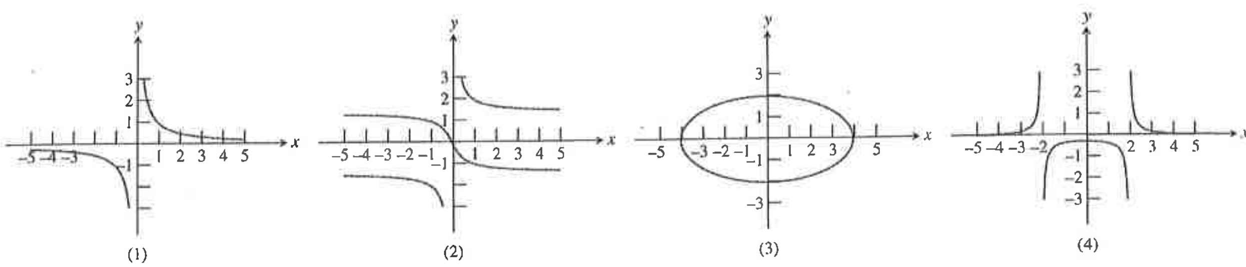


FIGURE 1.66 (Example 10)

A *function* whose inverse is a function has a graph that passes both the horizontal and vertical line tests (such as graph (1) in Example 10). Such a function is **one-to-one**, since every x is paired with a unique y and every y is paired with a unique x.

CAUTION ABOUT FUNCTION NOTATION

The symbol f^{-1} is read "f inverse" and should never be confused with the reciprocal of f. If f is a *function*, the symbol f^{-1} can *only* mean f inverse. The reciprocal of f must be written as $1/f$.

Definition Inverse Function

If f is a one-to-one function with domain D and range R, then the **inverse function of f**, denoted f^{-1}, is the function with domain R and range D defined by

$$f^{-1}(b) = a \quad \text{if and only if} \quad f(a) = b.$$

EXAMPLE 11 Finding an inverse function algebraically

Find an equation for $f^{-1}(x)$ if $f(x) = x/(x + 1)$.

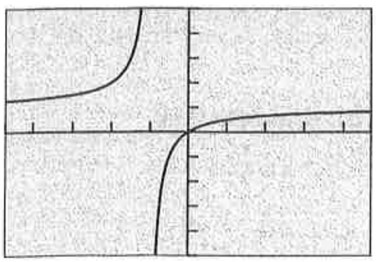


FIGURE 1.67 The graph of $f(x) = x/(x + 1)$. (Example 11)

SOLUTION The graph of f in Figure 1.67 suggests that f is one-to-one. The original function satisfies the equation $y = x/(x + 1)$. If f truly is one-to-one, the inverse function f^{-1} will satisfy the equation $x = y/(y + 1)$. (Note that we just switch the x and the y.)

If we solve this new equation for y we will have a formula for $f^{-1}(x)$:

$$x = \frac{y}{y+1}$$

$$x(y + 1) = y \qquad \text{Multiply by } y + 1.$$

$$xy + x = y \qquad \text{Distributive property}$$

$$xy - y = -x \qquad \text{Isolate the } y \text{ terms.}$$

$$y(x - 1) = -x \qquad \text{Factor out } y.$$

$$y = \frac{-x}{x-1} \qquad \text{Divide by } x - 1.$$

$$y = \frac{x}{1-x} \qquad \text{Multiply numerator and denominator by } -1.$$

Therefore $f^{-1}(x) = x/(1 - x)$. Now try Exercise 45.

Let us candidly admit two things regarding Example 11 before moving on to a graphical model for finding inverses. First, many functions are not one-to-one and so do not have inverse functions. Second, the algebra involved in finding an inverse function in the manner of Example 11 can be extremely difficult. We will actually find very few inverses this way. As you will learn in future chapters, we will usually rely on our understanding of how f maps x to y to understand how f^{-1} maps y to x.

It is possible to use the graph of f to produce a graph of f^{-1} without doing any algebra at all, thanks to the following geometric reflection property:

The Inverse Reflection Principle

The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line $y = x$. The points (a, b) and (b, a) are **reflections** of each other across the line $y = x$.

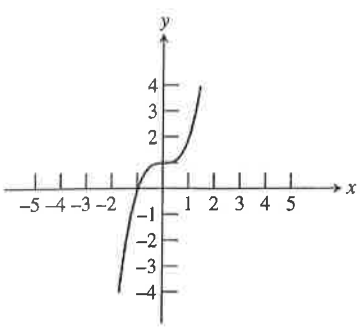


FIGURE 1.68 The graph of a one-to-one function. (Example 12)

EXAMPLE 12 Finding an inverse function graphically

The graph of a function $y = f(x)$ is shown in Figure 1.68. Sketch a graph of the function $y = f^{-1}(x)$. Is f a one-to-one function?

SOLUTION We need not find a formula for $f^{-1}(x)$. All we need to do is to find the reflection of the given graph across the line $y = x$. This can be done geometrically.

Imagine a mirror along the line $y = x$ and draw the reflection of the given graph in the mirror. (See Figure 1.69.)

Another way to visualize this process is to imagine the graph to be drawn on a large pane of glass. Imagine the glass rotating around the line $y = x$ so that the *positive* x-axis switches places with the *positive* y-axis. (The back of the glass must be rotated to the front for this to occur.) The graph of f will then become the graph of f^{-1}.

Since the inverse of f has a graph that passes the horizontal and vertical line test, f is a one-to-one function. Now try Exercise 53.

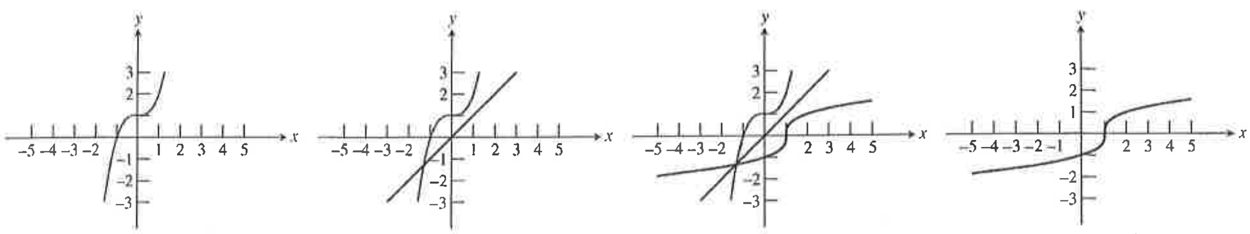


FIGURE 1.69 The Mirror Method. The graph of f is reflected in an imaginary mirror along the line $y = x$ to produce the graph of f^{-1}. (Example 12)

There is a natural connection between inverses and function composition that gives further insight into what an inverse actually does: It "undoes" the action of the original function. This leads to the following rule:

The Inverse Composition Rule

A function f is one-to-one with inverse function g if and only if

$f(g(x)) = x$ for every x in the domain of g, and

$g(f(x)) = x$ for every x in the domain of f.

EXAMPLE 13 Verifying inverse functions

Show algebraically that $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$ are inverse functions.

SOLUTION We use the Inverse Composition Rule.

$$f(g(x)) = f(\sqrt[3]{x - 1}) = (\sqrt[3]{x - 1})^3 + 1 = x - 1 + 1 = x$$

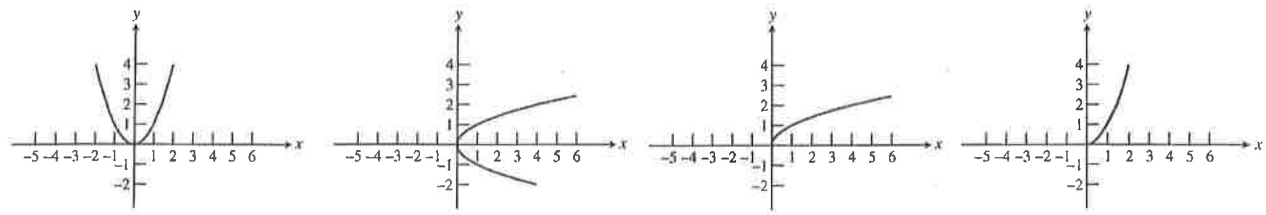
$$g(f(x)) = g(x^3 + 1) = \sqrt[3]{(x^3 + 1) - 1} = \sqrt[3]{x^3} = x$$

Since these equations are true for all x, the Inverse Composition Rule guarantees that f and g are inverses.

You do not have far to go to find graphical support of this algebraic verification, since these are the functions whose graphs are shown in Example 12!

Now try Exercise 57.

Some functions are so important that we need to study their inverses even though they are not one-to-one. A good example is the square root function, which is the "inverse" of the square function. It is not the inverse of the *entire* squaring function, because the full parabola fails the horizontal line test. Figure 1.70 shows that the function $y = \sqrt{x}$ is really the inverse of a "restricted-domain" version of $y = x^2$ defined only for $x \geq 0$.



The graph of $y = x^2$ (not one-to-one). The inverse relation of $y = x^2$ (not a function). The graph of $y = \sqrt{x}$ (a function). The graph of the function whose inverse is $y = \sqrt{x}$.

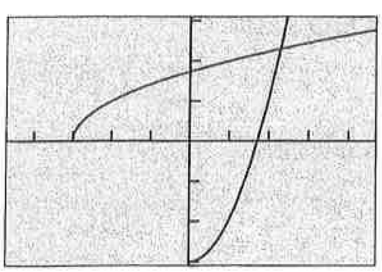
FIGURE 1.70 The function $y = x^2$ has no inverse function, but $y = \sqrt{x}$ is the inverse function of $y = x^2$ on the restricted domain $[0, \infty)$.

The consideration of domains adds a refinement to the algebraic inverse-finding method of Example 11, which we now summarize:

How to Find an Inverse Function Algebraically

Given a formula for a function f, proceed as follows to find a formula for f^{-1}.

1. Determine that there is a function f^{-1} by checking that f is one-to-one. State any restrictions on the domain of f. (Note that it might be necessary to impose some to get a one-to-one version of f.)
2. Switch x and y in the formula $y = f(x)$.
3. Solve for y to get the formula $y = f^{-1}(x)$. State any restrictions on the domain of f^{-1}.



[-4.7, 4.7] by [-3.1, 3.1]

FIGURE 1.71 The graph of $f(x) = \sqrt{x+3}$ and its inverse, a restricted version of $y = x^2 - 3$. (Example 14)

EXAMPLE 14 Finding an inverse function

Show that $f(x) = \sqrt{x+3}$ has an inverse function and find a rule for $f^{-1}(x)$. State any restrictions on the domains of f and f^{-1}.

SOLUTION

Solve Algebraically

The graph of f passes the horizontal line test, so f has an inverse function (Figure 1.71). Note that f has domain $[-3, \infty)$ and range $[0, \infty)$.

To find f^{-1} we write

$y = \sqrt{x+3}$	where $x \geq -3, y \geq 0$	
$x = \sqrt{y+3}$	where $y \geq -3, x \geq 0$	Interchange x and y.
$x^2 = y + 3$	where $y \geq -3, x \geq 0$	Square.
$y = x^2 - 3$	where $y \geq -3, x \geq 0$	Solve for y.

Thus $f^{-1}(x) = x^2 - 3$, with an "inherited" domain restriction of $x \geq 0$. Figure 1.71 shows the two functions. Note the domain restriction of $x \geq 0$ imposed on the parabola $y = x^2 - 3$.

Support Graphically

Use a grapher in parametric mode and compare the graphs of the two sets of parametric equations with Figure 1.71:

$$x = t \quad y = \sqrt{t+3} \qquad \text{and} \qquad x = \sqrt{t+3} \quad y = t$$

Now try Exercise 47.

QUICK REVIEW 1.4

(For help, go to Sections P.3 and P.4.)

In Exercises 1–10, solve the equation for y.

1. $x = 3y - 6$
2. $x = 0.5y + 1$
3. $x = y^2 + 4$
4. $x = y^2 - 6$
5. $x = \dfrac{y-2}{y+3}$
6. $x = \dfrac{3y-1}{y+2}$
7. $x = \dfrac{2y+1}{y-4}$
8. $x = \dfrac{4y+3}{3y-1}$
9. $x = \sqrt{y+3},\ y \geq -3$
10. $x = \sqrt{y-2},\ y \geq 2$

SECTION 1.4 EXERCISES

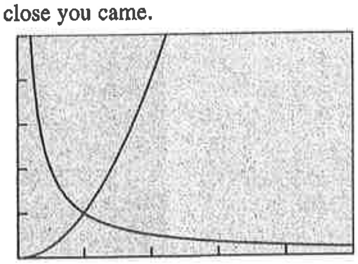
In Exercises 1–4, find formulas for the functions $f + g, f - g$, and fg. Give the domain of each. Also state the domain of each function.

1. $f(x) = 2x - 1;\ g(x) = x^2$
2. $f(x) = (x-1)^2;\ g(x) = 3 - x$
3. $f(x) = \sqrt{x};\ g(x) = \sin x$
4. $f(x) = \sqrt{x+5};\ g(x) = |x+3|$

In Exercises 5 and 6, find formulas for f/g and g/f. Give the domain of each. Also state the domain of each function.

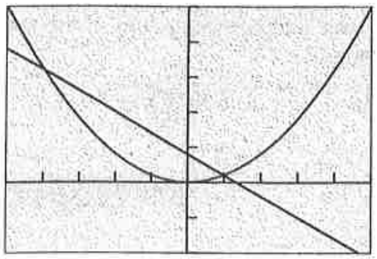
5. $f(x) = \sqrt{x+3};\ g(x) = x^2$
6. $f(x) = \sqrt{x-2};\ g(x) = \sqrt{x+4}$

7. $f(x) = x^2$ and $g(x) = 1/x$ are shown below in the viewing window [0, 5] by [0, 5]. Sketch the graph of the sum $(f+g)(x)$ by adding the y-coordinates directly from the graphs. Then graph the sum on your calculator and see how close you came.



[0, 5] by [0, 5]

8. The graphs of $f(x) = x^2$ and $g(x) = 4 - 3x$ are shown below in the viewing window $[-5, 5]$ by $[-10, 25]$. Sketch the graph of the difference $(f - g)(x)$ by subtracting the y-coordinates directly from the graphs. Then graph the difference on your calculator and see how close you came.



$[-5, 5]$ by $[-10, 25]$

In Exercises 9 and 10, find $(f \circ g)(3)$ and $(g \circ f)(-2)$

9. $f(x) = 2x - 3$; $g(x) = x + 1$

10. $f(x) = x^2 - 1$; $g(x) = 2x - 3$

In Exercises 11–14, find $f(g(x))$ and $g(f(x))$. State the domain of each.

11. $f(x) = 3x + 2$; $g(x) = x - 1$

12. $f(x) = x^2 - 1$; $g(x) = \dfrac{1}{x - 1}$

13. $f(x) = x^2 - 2$; $g(x) = \sqrt{x + 1}$

14. $f(x) = \dfrac{1}{x - 1}$; $g(x) = \sqrt{x}$

In Exercises 15–20, find $f(x)$ and $g(x)$ so that the function can be described as $y = f(g(x))$.

15. $y = \sqrt{x^2 - 5x}$

16. $y = (x^3 + 1)^2$

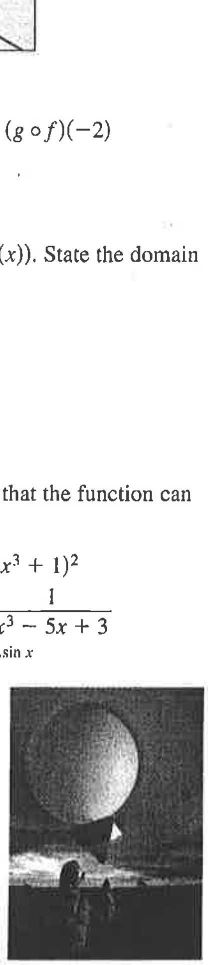
17. $y = |3x - 2|$

18. $y = \dfrac{1}{x^3 - 5x + 3}$

19. $y = (x - 3)^5 + 2$

20. $y = e^{\sin x}$

21. Weather Balloons A high-altitude spherical weather balloon expands as it rises due to the drop in atmospheric pressure. Suppose that the radius r increases at the rate of 0.03 inches per second and that $r = 48$ inches at time $t = 0$. Determine an equation that models the volume V of the balloon at time t and find the volume when $t = 300$ seconds.



22. A Snowball's Chance Jake stores a small cache of 4-inch diameter snowballs in the basement freezer, unaware that the freezer's self-defrosting feature will cause each snowball to lose about 1 cubic inch of volume every 40 days. He remembers them a year later (call it 360 days) and goes to retrieve them. What is their diameter then?

23. Satellite Photography A satellite camera takes a rectangular-shaped picture. The smallest region that can be photographed is a 5-km by 7-km rectangle. As the camera zooms out, the length l and width w of the rectangle increase at a rate of 2 km/sec. How long does it take for the area A to be at least 5 times its original size?

24. Computer Imaging New Age Special Effects, Inc., prepares computer software based on specifications prepared by film directors. To simulate an approaching vehicle, they begin with a computer image of a 5-cm by 7-cm by 3-cm box. The program increases each dimension at a rate of 2 cm/sec. How long does it take for the volume V of the box to be at least 5 times its initial size?

25. Which of the ordered pairs $(1, 1)$, $(4, -2)$, and $(3, -1)$ are in the relation given by $3x + 4y = 5$?

26. Which of the ordered pairs $(5, 1)$, $(3, 4)$, and $(0, -5)$ are in the relation given by $x^2 + y^2 = 25$?

In Exercises 27–30, find two functions defined implicitly by each given relation.

27. $x^2 + y^2 = 25$

28. $x + y^2 = 25$

29. $x^2 - y^2 = 25$

30. $3x^2 - y^2 = 25$

In Exercises 31–34, find the (x, y) pair for the value of the parameter.

31. $x = 3t$ and $y = t^2 + 5$ for $t = 2$

32. $x = 5t - 7$ and $y = 17 - 3t$ for $t = -2$

33. $x = t^3 - 4t$ and $y = \sqrt{t + 1}$ for $t = 3$

34. $x = |t + 3|$ and $y = 1/t$ for $t = -8$

In Exercises 35–38, complete the following. (a) Find the points determined by $t = -3, -2, -1, 0, 1, 2$, and 3. (b) Find a direct relationship between y and x and determine whether the parametric equations determine y as a function of x. (c) Graph the relationship in the xy-plane.

35. $x = 2t$ and $y = 3t - 1$

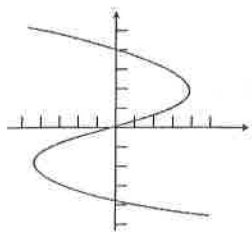
36. $x = t + 1$ and $y = t^2 - 2t$

37. $x = t^2$ and $y = t - 2$

38. $x = \sqrt{t}$ and $y = 2t - 5$

In Exercises 39–42, the graph of a relation is shown. **(a)** Is the relation a function? **(b)** Does the relation have a inverse that is a function?

39.



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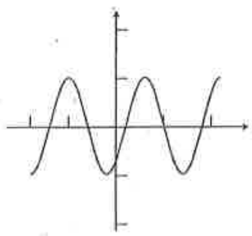

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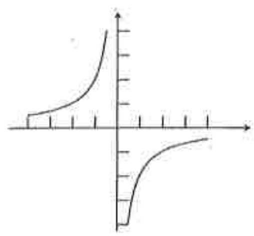

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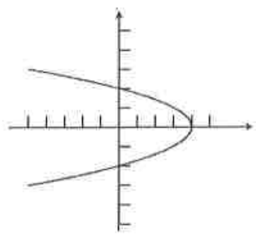


In Exercises 43–52, find a formula for $f^{-1}(x)$. Give the domain of f^{-1}, including any restrictions "inherited" from f.

43. $f(x) = 3x - 6$

44. $f(x) = 2x + 5$

45. $f(x) = \dfrac{2x - 3}{x + 1}$

46. $f(x) = \dfrac{x + 3}{x - 2}$

47. $f(x) = \sqrt{x - 3}$

48. $f(x) = \sqrt{x + 2}$

49. $f(x) = x^3$

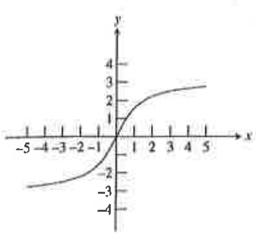
50. $f(x) = x^3 + 5$

51. $f(x) = \sqrt[3]{x + 5}$

52. $f(x) = \sqrt[3]{x - 2}$

In Exercises 53–56, determine whether the function is one-to-one. If it is one-to-one, sketch the graph of the inverse.

53.



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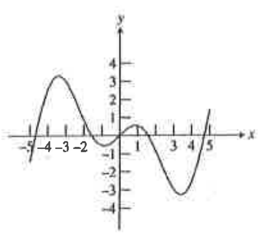

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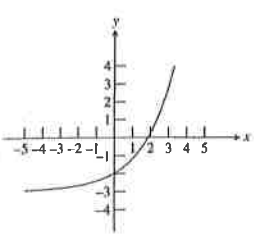

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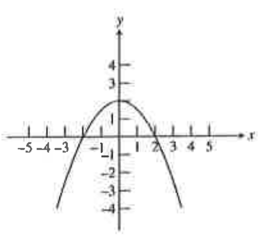


In Exercises 57–62, confirm that f and g are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

57. $f(x) = 3x - 2$ and $g(x) = \dfrac{x + 2}{3}$

58. $f(x) = \dfrac{x + 3}{4}$ and $g(x) = 4x - 3$

59. $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x - 1}$

60. $f(x) = \dfrac{7}{x}$ and $g(x) = \dfrac{7}{x}$

61. $f(x) = \dfrac{x + 1}{x}$ and $g(x) = \dfrac{1}{x - 1}$

62. $f(x) = \dfrac{x + 3}{x - 2}$ and $g(x) = \dfrac{2x + 3}{x - 1}$

63. Currency Conversion In May of 2002 the exchange rate for converting U.S. dollars (x) to euros (y) was $y = 1.08x$.

(a) How many euros could you get for \$100 U.S.?

(b) What is the inverse function, and what conversion does it represent?

(c) In the spring of 2002, a tourist had an elegant lunch in Provence, France ordering from a "fixed price" 48-euro menu. How much was that in U.S. dollars?

64. Temperature Conversion The formula for converting Celsius temperature (x) to Kelvin temperature is $k(x) = x + 273.16$. The formula for converting Fahrenheit temperature (x) to Celsius temperature is $c(x) = (5/9)(x - 32)$.

(a) Find a formula for $c^{-1}(x)$. What is this formula used for?

(b) Find $(k \circ c)(x)$. What is this formula used for?

Standardized Test Questions

65. True or False The domain of the quotient function $(f/g)(x)$ consists of all numbers that belong to both the domain of f and the domain of g. Justify your answer.

66. True or False The domain of the product function $(fg)(x)$ consists of all numbers that belong to either the domain of f or the domain of g. Justify your answer.

In Exercises 67–70, answer the question without using a calculator.

67. Multiple Choice Which ordered pair is in the *inverse* of the relation given by $x^2y + 5y = 9$?

(a) (2, 1) **(b)** (–2, 1)

(c) (–1, 2) **(d)** (2, –1)

(e) (1, –2)

2y + xy
y(2+x)

68. Multiple Choice Which ordered pair is not in the *inverse* of the relation given by $xy^2 - 3x = 12$?

(a) (0, –4)

(b) (4, 1)

(c) (3, 2)

(d) (2, 12)

(e) (1, –6)

69. Multiple Choice Which function is the *inverse* of the function $f(x) = 3x - 2$?

(a) $g(x) = \frac{x}{3} + 2$

(b) $g(x) = 2 - 3x$

(c) $g(x) = \frac{x+2}{3}$

(d) $g(x) = \frac{x-3}{2}$

(e) $g(x) = \frac{x-2}{3}$

70. Multiple Choice Which function is the *inverse* of the function $f(x) = x^3 + 1$?

(a) $g(x) = \sqrt[3]{x-1}$

(b) $g(x) = \sqrt[3]{x} - 1$

(c) $g(x) = x^3 - 1$

(d) $g(x) = \sqrt[3]{x+1}$

(e) $g(x) = 1 - x^3$

Explorations

71. Function Properties Inherited by Inverses There are some properties of functions that are automatically shared by inverse functions (when they exist) and some that are not. Suppose that f has an inverse function f^{-1}. Give an algebraic or graphical argument (not a rigorous formal proof) to show that each of these properties of f must necessarily be shared by f^{-1}.

(a) f is continuous.

(b) f is one-to-one.

(c) f is odd (graphically, symmetric with respect to the origin).

(d) f is increasing.

72. Function Properties not Inherited by Inverses There are some properties of functions that are not necessarily shared by inverse functions, even if the inverses exist. Suppose that f has an inverse function f^{-1}. For each of the following properties, give an example to show that f can have the property while f^{-1} does not.

(a) f has a graph with a horizontal asymptote.

(b) f has domain all real numbers.

(c) f has a graph that is bounded above.

(d) f has a removable discontinuity at $x = 5$.

73. Scaling Algebra Grades A teacher gives a challenging algebra test to her class. The lowest score is 52, which she decides to scale to 70. The highest score is 88, which she decides to scale to 97.

(a) Using the points (52, 70) and (88, 97), find a linear equation that can be used to convert raw scores to scaled grades.

(b) Find the inverse of the function defined by this linear equation. What does the inverse function do?

74. Writing to Learn (Continuation of Exercise 73) Explain why it is important for fairness that the scaling function used by the teacher be an *increasing* function. (Caution: It is *not* because "everyone's grade must go up." What would the scaling function in Exercise 73 do for a student who does enough "extra credit" problems to get a raw score of 136?)

Extending the Ideas

75. Modeling a Fly Ball Parametrically A baseball that leaves the bat at an angle of 60° from horizontal traveling 110 feet per second follows a path that can be modeled by the following pair of parametric equations. (You might enjoy verifying this if you have studied motion in physics.)

$x = 110(t)\cos(60°)$

$y = 110(t)\sin(60°) - 16t^2$

You can simulate the flight of the ball on a grapher. Set your grapher to parametric mode and put the functions above in for X2T and Y2T. Set X1T = 325 and Y1T = 5T to draw a 30-foot fence 325 feet from home plate. Set Tmin = 0, Tmax = 6, Tstep = 0.1, Xmin = 0, Xmax = 350, Xscl = 0, Ymin = 0, Ymax = 300, and Yscl = 0.

(a) Now graph the function. Does the fly ball clear the fence?

(b) Change the angle to 30° and run the simulation again. Does the ball clear the fence?

(c) What angle is optimal for hitting the ball? Does it clear the fence when hit at that angle?

76. The Baylor GPA Scale Revisited *(See Problem 78 in Section 1.2.)* The function used to convert Baylor School percentage grades to GPAs on a 4-point scale is

$$y = \left(\frac{3^{1.7}}{30}(x - 65)\right)^{\frac{1}{1.7}} + 1.$$

The function has domain [65, 100]. Anything below 65 is a failure and automatically converts to a GPA of 0.

(a) Find the inverse function algebraically. What can the inverse function be used for?

(b) Does the inverse function have any domain restrictions?

(c) Verify with a graphing calculator that the function found in (a) and the given function are really inverses.

77. Group Activity (Continuation of Exercise 76) The number 1.7 that appears in two places in the GPA scaling formula is called the scaling factor (k). The value of k can be changed to alter the curvature of the graph while keeping the points (1, 65) and (95, 4) fixed. It was felt that the lowest D (65) needed to be scaled to 1.0, while the middle A (95) needed to be scaled to 4.0. The faculty's Academic Council considered several values of k before settling on 1.7 as the number that gives the "fairest" GPAs for the other percentage grades.

Try changing k to other values between 1 and 2. What kind of scaling curve do you get when $k = 1$? Do you agree with the Baylor decision that $k = 1.7$ gives the fairest GPAs?

78. Revisiting Example 6 Solve $x^2y + y^2 = 5$ for y using the quadratic formula and graph the pair of implicit functions.

1.5 GRAPHICAL TRANSFORMATIONS

What you'll learn about

- Transformations
- Vertical and Horizontal Translations
- Reflections Across Axes
- Vertical and Horizontal Stretches and Shrinks
- Combining Transformations

. . . and why

Studying transformations will help you to understand the relationships between graphs that have similarities but are not the same.

Transformations

The following functions are all different:

$$y = x^2$$
$$y = (x - 3)^2$$
$$y = 1 - x^2$$
$$y = x^2 - 4x + 5$$

However, a look at their graphs shows that, while no two are exactly the same, all four have the same identical *shape* and *size*. Understanding how algebraic alterations change the shapes, sizes, positions, and orientations of graphs is helpful for understanding the connection between algebraic and graphical models of functions.

In this section we relate graphs using **transformations**, which are functions that map real numbers to real numbers. By acting on the x-coordinates and y-coordinates of points, transformations change graphs in predictable ways. **Rigid transformations**, which leave the size and shape of a graph unchanged, include horizontal translations, vertical translations, reflections, or any combination of these. **Non-rigid transformations**, which generally distort the shape of a graph, include horizontal or vertical stretches and shrinks.

Vertical and Horizontal Translations

A **vertical translation** of the graph of $y = f(x)$ is a shift of the graph up or down in the coordinate plane. A **horizontal translation** is a shift of the graph to the left or the right. The following exploration will give you a good feel for what translations are and how they occur.

TECHNOLOGY ALERT

In Exploration 1, the notation $y_1(x + 3)$ means the function y_1, evaluated at $x + 3$. It does not mean multiplication.

EXPLORATION 1 Introducing Translations

Set your viewing window to $[-5, 5]$ by $[-5, 15]$ and your graphing mode to sequential as opposed to simultaneous.

1. Graph the functions

$$y_1 = x^2 \qquad y_4 = y_1(x) - 2 = x^2 - 2$$
$$y_2 = y_1(x) + 3 = x^2 + 3 \qquad y_5 = y_1(x) - 4 = x^2 - 4$$
$$y_3 = y_1(x) + 1 = x^2 + 1$$

on the same screen. What effect do the $+3$, $+1$, -2, and -4 seem to have?

2. Graph the functions

$$y_1 = x^2 \qquad y_4 = y_1(x - 2) = (x - 2)^2$$
$$y_2 = y_1(x + 3) = (x + 3)^2 \qquad y_5 = y_1(x - 4) = (x - 4)^2$$
$$y_3 = y_1(x + 1) = (x + 1)^2$$

on the same screen. What effect do the $+3$, $+1$, -2, and -4 seem to have?

3. Repeat steps 1 and 2 for the functions $y_1 = x^3$, $y_1 = |x|$, and $y_1 = \sqrt{x}$. Do your observations agree with those you made after steps 1 and 2?

In general, *replacing* x by $x - c$ shifts the graph horizontally c units. Similarly, *replacing* y by $y - c$ shifts the graph vertically c units. If c is positive the shift is to the right or up; if c is negative the shift is to the left or down.

This is a nice, consistent rule that unfortunately gets complicated by the fact that the c for a vertical shift rarely shows up being subtracted from y. Instead, it usually shows up on the other side of the equal sign being *added* to $f(x)$. That leads us to the following rule, which only *appears* to be different for horizontal and vertical shifts:

Translations

Let c be a positive real number. Then the following transformations result in translations of the graph of $y = f(x)$:

Horizontal translations

$y = f(x - c)$	a translation to the right by c units
$y = f(x + c)$	a translation to the left by c units

Vertical translations

$y = f(x) + c$	a translation up by c units
$y = f(x) - c$	a translation down by c units

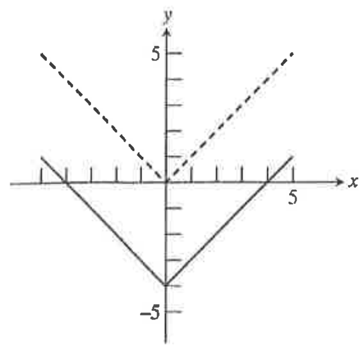


FIGURE 1.72 $y = |x| - 4$ (Example 1)

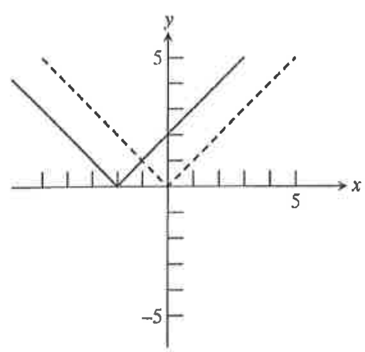


FIGURE 1.73 $y = |x + 2|$ (Example 1)

EXAMPLE 1 Vertical translations

Describe how the graph of $y = |x|$ can be transformed to the graph of the given equation.

(a) $y = |x| - 4$ **(b)** $y = |x + 2|$

SOLUTION

(a) The equation is in the form $y = f(x) - 4$, a translation down by 4 units. See Figure 1.72.

(b) The equation is in the form $y = f(x + 2)$, a translation left by 2 units. See Figure 1.73. Now try Exercise 3.

EXAMPLE 2 Finding equations for translations

Each view in Figure 1.74 shows the graph of $y_1 = x^3$ and a vertical or horizontal translation y_2. Write an equation for y_2 as shown in each graph.

SOLUTION

(a) $y_2 = x^3 - 3 = y_1(x) - 3$ (a vertical translation down by 3 units)

(b) $y_2 = (x + 2)^3 = y_1(x + 2)$ (a horizontal translation left by 2 units)

(c) $y_2 = (x - 3)^3 = y_1(x - 3)$ (a horizontal translation right by 3 units)

Now try Exercise 25.

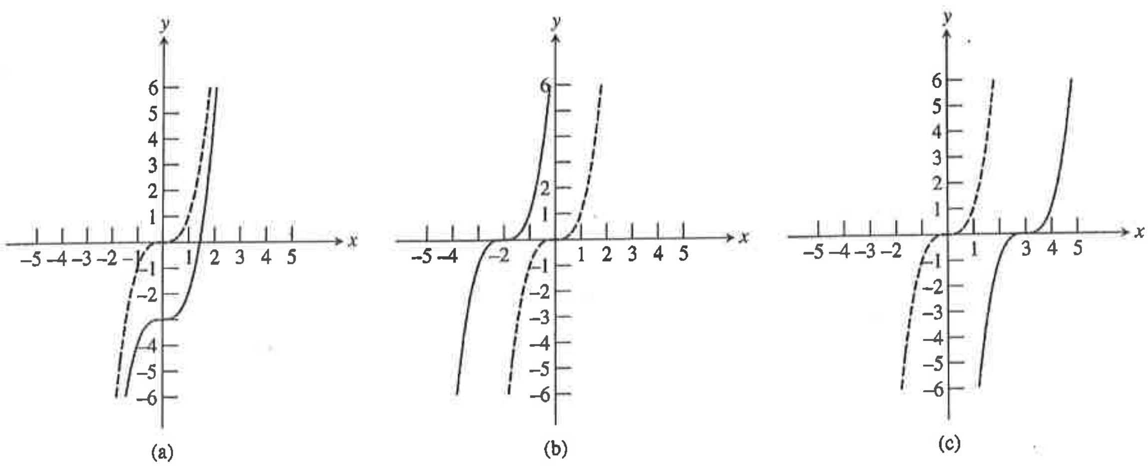


FIGURE 1.74 Translations of $y_1 = x^3$. (Example 2)

Reflections Across Axes

Points (x, y) and $(x, -y)$ are **reflections of each other across the x-axis**. Points (x, y) and $(-x, y)$ are **reflections of each other across the y-axis**. (See Figure 1.75.) Two points (or graphs) that are symmetric with respect to a line are **reflections of each other across that line**.

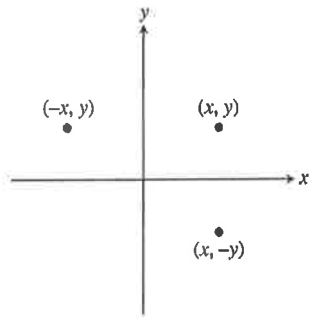


FIGURE 1.75 The point (x, y) and its reflections across the x- and y-axes.

EXPLORATION 2 | Introducing Reflections

Set your viewing window to $[-5, 5]$ by $[-4, 4]$ and your graphing mode to sequential as opposed to simultaneous.

1. Graph the functions $y_1 = (x - 2)^2 + 1$ and $y_2 = -y_1(x)$ on the same screen. What is the geometrical relationship between the graphs?
2. Graph the functions $y_1 = (x - 2)^2 + 1$ and $y_2 = y_1(-x)$ on the same screen. What is the geometrical relationship between the graphs?
3. Graph the functions $y_1 = \sqrt{x + 2} + 1$ and $y_2 = -y_1(x)$ on the same screen. What is the geometrical relationship between the graphs?
4. Graph the functions $y_1 = \sqrt{x + 2} + 1$ and $y_2 = y_1(-x)$ on the same screen. What is the geometrical relationship between the graphs?
5. Graph the functions $y_1 = x^3 - 3x^2 + x + 1$ and $y_2 = -y_1(x)$ on the same screen. What is the geometrical relationship between the graphs?
6. Graph the functions $y_1 = x^3 - 3x^2 + x + 1$ and $y_2 = y_1(-x)$ on the same screen. What is the geometrical relationship between the graphs?

Exploration 2 suggests that a reflection across the x-axis results when y is replaced by $-y$, and a reflection across the y-axis results when x is replaced by $-x$. This should make sense to you if you understand Figure 1.75.

Reflections

The following transformations result in reflections of the graph of $y = f(x)$:

Across the x-axis

$$y = -f(x)$$

Across the y-axis

$$y = f(-x)$$

EXAMPLE 3 Finding equations for reflections

Find an equation for the reflection of $f(x) = \dfrac{5x - 9}{x^2 + 3}$ across each axis.

SOLUTION

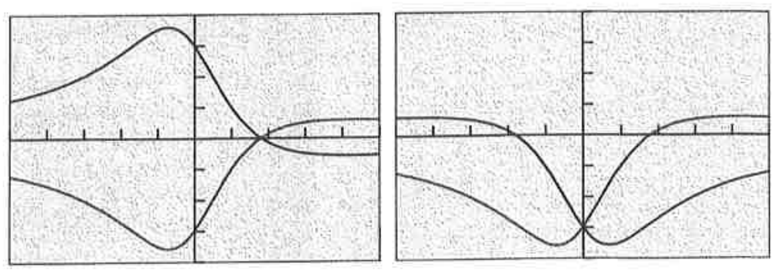
Solve Algebraically

Across the x-axis: $y = -f(x) = -\dfrac{5x - 9}{x^2 + 3} = \dfrac{9 - 5x}{x^2 + 3}$

Across the y-axis: $y = f(-x) = \dfrac{5(-x) - 9}{(-x)^2 + 3} = \dfrac{-5x - 9}{x^2 + 3}$

Support Graphically

The graphs in Figure 1.76 support our algebraic work.



[–5, 5] by [–4, 4]
(a)

[–5, 5] by [–4, 4]
(b)

FIGURE 1.76 Reflections of $f(x) = (5x - 9)/(x^2 + 3)$ across (a) the x-axis and (b) the y-axis. (Example 3)

Now try Exercise 29

You might expect that odd and even functions, whose graphs already possess special symmetries, would exhibit special behavior when reflected across the axes. They do, as shown by Example 4 and Exercises 33–34.

EXAMPLE 4 Reflecting even functions

Prove that the graph of an even function remains unchanged when it is reflected across the y-axis.

SOLUTION

Note that we can get plenty of graphical support for these statements by reflecting the graphs of various even functions, but what is called for here is **proof**, which will require algebra.

Let f be an even function; that is, $f(-x) = f(x)$ for all x in the domain of f. To reflect the graph of $y = f(x)$ across the y-axis, we make the transformation $y = f(-x)$. But $f(-x) = f(x)$ for all x in the domain of f, so this transformation results in $y = f(x)$. The graph of f therefore remains unchanged.

Now try Exercise 33.

Vertical and Horizontal Stretches and Shrinks

We now investigate what happens when we multiply all the y-coordinates (or all the x-coordinates) of a graph by a fixed real number.

EXPLORATION 3 Introducing Stretches and Shrinks

Set your viewing window to $[-4.7, 4.7]$ by $[-1.1, 5.1]$ and your graphing mode to sequential as opposed to simultaneous.

1. Graph the functions

$$y_1 = \sqrt{4 - x^2}$$
$$y_2 = 1.5y_1(x) = 1.5\sqrt{4 - x^2}$$
$$y_3 = 2y_1(x) = 2\sqrt{4 - x^2}$$
$$y_4 = 0.5y_1(x) = 0.5\sqrt{4 - x^2}$$
$$y_5 = 0.25y_1(x) = 0.25\sqrt{4 - x^2}$$

on the same screen. What effect do the 1.5, 2, 0.5, and 0.25 seem to have?

2. Graph the functions

$$y_1 = \sqrt{4 - x^2}$$
$$y_2 = y_1(1.5x) = \sqrt{4 - (1.5x)^2}$$
$$y_3 = y_1(2x) = \sqrt{4 - (2x)^2}$$
$$y_4 = y_1(0.5x) = \sqrt{4 - (0.5x)^2}$$
$$y_5 = y_1(0.25x) = \sqrt{4 - (0.25x)^2}$$

on the same screen. What effect do the 1.5, 2, 0.5, and 0.25 seem to have?

Exploration 3 suggests that multiplication of x or y by a constant results in a horizontal or vertical stretching or shrinking of the graph.

In general, *replacing x by x/c* distorts the graph horizontally by a factor of c. Similarly, *replacing y by y/c* distorts the graph vertically by a factor of c. If c is greater than 1 the distortion is a stretch; if c is less than 1 the distortion is a shrink.

As with translations, this is a nice, consistent rule that unfortunately gets complicated by the fact that the c for a vertical stretch or shrink rarely shows up as a divisor of y. Instead, it usually shows up on the other side of the equal sign as a *factor* multiplied by $f(x)$. That leads us to the following rule:

Stretches and Shrinks

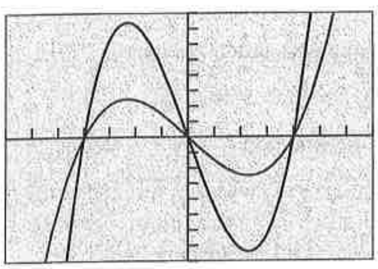
Let c be a positive real number. Then the following transformations result in stretches or shrinks of the graph of $y = f(x)$:

Horizontal stretches or shrinks

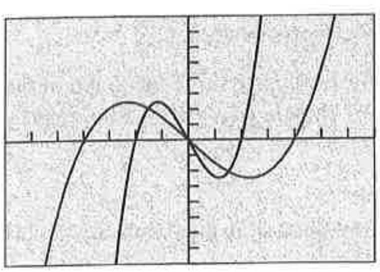
$$y = f\left(\frac{x}{c}\right) \quad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$

Vertical stretches or shrinks

$$y = c \cdot f(x) \quad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$



[–7, 7] by [–80, 80]
(a)



[–7, 7] by [–80, 80]
(b)

FIGURE 1.77 The graph of $y_1 = f(x) = x^3 - 16x$, shown with (a) a vertical stretch and (b) a horizontal shrink. (Example 5)

EXAMPLE 5 Finding equations for stretches and shrinks

Let C_1 be the curve defined by $y_1 = f(x) = x^3 - 16x$. Find equations for the following non-rigid transformations of C_1:

(a) C_2 is a vertical stretch of C_1 by a factor of 3.

(b) C_3 is a horizontal shrink of C_1 by a factor of 1/2.

SOLUTION

Solve Algebraically

(a) Denote the equation for C_2 by y_2. Then

$$\begin{aligned} y_2 &= 3 \cdot f(x) \\ &= 3(x^3 - 16x) \\ &= 3x^3 - 48x \end{aligned}$$

(b) Denote the equation for C_3 by y_3. Then

$$\begin{aligned} y_3 &= f\left(\frac{x}{1/2}\right) \\ &= f(2x) \\ &= (2x)^3 - 16(2x) \\ &= 8x^3 - 32x \end{aligned}$$

Support Graphically

The graphs in Figure 1.77 support our algebraic work.

Now try Exercise 39.

Combining Transformations

Transformations may be performed in succession—one after another. If the transformations include stretches, shrinks, or reflections, the order in which the transformations are performed may make a difference. In those cases, be sure to pay particular attention to order.

EXAMPLE 6 Combining transformations in order

(a) The graph of $y = x^2$ undergoes the following transformations, in order. Find the equation of the graph that results.

- a horizontal shift 2 units to the right
- a vertical stretch by a factor of 3
- a vertical translation 5 units up

(b) Apply the transformations in (a) in the opposite order and find the equation of the graph that results.

SOLUTION

(a) Applying the transformations in order, we have

$$x^2 \Rightarrow (x-2)^2 \Rightarrow 3(x-2)^2 \Rightarrow 3(x-2)^2 + 5$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 17$.

(b) Applying the transformations in the opposite order, we have

$$x^2 \Rightarrow x^2 + 5 \Rightarrow 3(x^2 + 5) \Rightarrow 3((x-2)^2 + 5)$$

Expanding the final expression, we get the function $y = 3x^2 - 12x + 27$.

The second graph is ten units higher than the first graph because the vertical stretch lengthens the vertical translation when the translation occurs first. Order often matters when stretches, shrinks, or reflections are involved.

Now try Exercise 47.

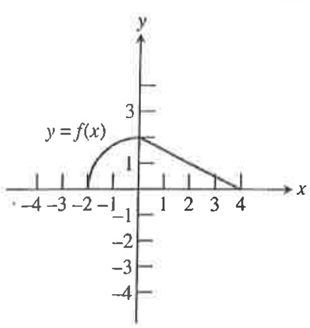


FIGURE 1.78 The graph of the function $y = f(x)$ in Example 7.

EXAMPLE 7 Transforming a graph geometrically

The graph of $y = f(x)$ is shown in Figure 1.78. Determine the graph of the composite function $y = 2f(x+1) - 3$ by showing the effect of a sequence of transformations on the graph of $y = f(x)$.

SOLUTION

The graph of $y = 2f(x+1) - 3$ can be obtained from the graph of $y = f(x)$ by the following sequence of transformations:

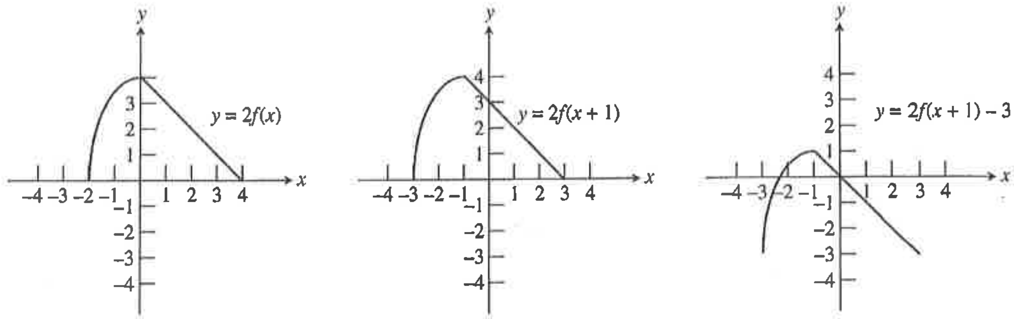
(a) a vertical stretch by a factor of 2 to get $y = 2f(x)$ (Figure 1.79a)

(b) a horizontal translation 1 unit to the left to get $y = 2f(x+1)$ (Figure 1.79b)

(c) a vertical translation 3 units down to get $y = 2f(x+1) - 3$ (Figure 1.79c)

(The order of the first two transformations can be reversed without changing the final graph.)

Now try Exercise 51.



Vertical stretch of factor 2

(a)

Horizontal translation left 1 unit

(b)

Vertical translation down 3 units

(c)

FIGURE 1.79 Transforming the graph of $y = f(x)$ in Figure 1.78 to get the graph of $y = 2f(x + 1) - 3$. (Example 7)

12 − 4 right y axis reflect

QUICK REVIEW 1.5

(For help, go to Section A.2.)

In Exercises 1–6, write the expression as a binomial squared.

1. $x^2 + 2x + 1$ **2.** $x^2 - 6x + 9$

3. $x^2 + 12x + 36$ **4.** $4x^2 + 4x + 1$

5. $x^2 - 5x + \frac{25}{4}$ **6.** $4x^2 - 20x + 25$

In Exercises 7–10, perform the indicated operations and simplify.

7. $(x - 2)^2 + 3(x - 2) + 4$

8. $2(x + 3)^2 - 5(x + 3) - 2$

9. $(x - 1)^3 + 3(x - 1)^2 - 3(x - 1)$

10. $2(x + 1)^3 - 6(x + 1)^2 + 6(x + 1) - 2$

SECTION 1.5 EXERCISES

In Exercises 1–8, describe how the graph of $y = x^2$ can be transformed to the graph of the given equation.

1. $y = x^2 - 3$ **2.** $y = x^2 + 5.2$

3. $y = (x + 4)^2$ **4.** $y = (x - 3)^2$

5. $y = (100 - x)^2$ **6.** $y = x^2 - 100$

7. $y = (x - 1)^2 + 3$ **8.** $y = (x + 50)^2 - 279$

In Exercises 9–12, describe how the graph of $y = \sqrt{x}$ can be transformed to the graph of the given equation.

9. $y = -\sqrt{x}$ **10.** $y = \sqrt{x - 5}$

11. $y = \sqrt{-x}$ **12.** $y = \sqrt{3 - x}$

In Exercises 13–16, describe how the graph of $y = x^3$ can be transformed to the graph of the given equation.

13. $y = 2x^3$ **14.** $y = (2x)^3$

15. $y = (0.2x)^3$ **16.** $y = 0.3x^3$

In Exercises 17–20, describe how to transform the graph of f into the graph of g.

17. $f(x) = \sqrt{x + 2}$ and $g(x) = \sqrt{x - 4}$

18. $f(x) = (x - 1)^2$ and $g(x) = -(x + 3)^2$

19. $f(x) = (x - 2)^3$ and $g(x) = -(x + 2)^3$

20. $f(x) = |2x|$ and $g(x) = 4|x|$

In Exercises 21–24, sketch the graphs of f, g, and h by hand. Support your answers with a grapher.

21. $f(x) = (x + 2)^2$
$g(x) = 3x^2 - 2$
$h(x) = -2(x - 3)^2$

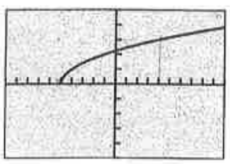
22. $f(x) = x^3 - 2$
$g(x) = (x + 4)^3 - 1$
$h(x) = 2(x - 1)^3$

23. $f(x) = \sqrt[3]{x + 1}$
$g(x) = 2\sqrt[3]{x} - 2$
$h(x) = -\sqrt[3]{x - 3}$

24. $f(x) = -2|x| - 3$
$g(x) = 3|x + 5| + 4$
$h(x) = |3x|$

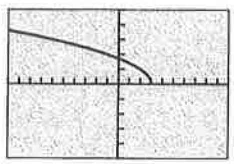
In Exercises 25–28, the graph is that of a function $y = f(x)$ that can be obtained by transforming the graph of $y = \sqrt{x}$. Write a formula for the function f.

25.



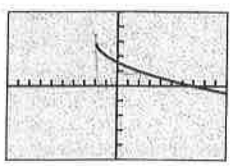
[–10, 10] by [–5, 5]

26.



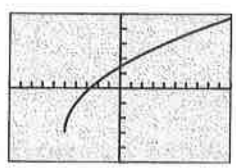
[–10, 10] by [–5, 5]

27.



[–10, 10] by [–5, 5]

28.



[–10, 10] by [–5, 5]
Vertical stretch = 2

In Exercises 29–32, find the equation of the reflection of f across **(a)** the x-axis and **(b)** the y-axis.

29. $f(x) = x^3 - 5x^2 - 3x + 2$ **30.** $f(x) = 2\sqrt{x + 3} - 4$

31. $f(x) = \sqrt[3]{8x}$ **32.** $f(x) = 3|x + 5|$

33. Reflecting Odd Functions Prove that the graph of an odd function is the same when reflected across the x-axis as it is when reflected across the y-axis.

34. Reflecting Odd Functions Prove that if an odd fuction is reflected about the y-axis and then reflected again about the x-axis, the result is the original function.

In Exercises 35–38, a graph of the given function can be obtained from $y = x^2$ by both a vertical stretch (or shrink) and by a horizontal shrink (or stretch). In each case, identify the shrink and stretch factors.

35. $y = 9x^2$ **36.** $y = \frac{1}{4}x^2$

37. $y = 16x^2$ **38.** $y = 5x^2$

In Exercises 39–42, transform the given function by **(a)** a vertical stretch by a factor of 2, and **(b)** a horizontal shrink by a factor of 1/3.

39. $f(x) = x^3 - 4x$ **40.** $f(x) = |x + 2|$

41. $f(x) = x^2 + x - 2$ **42.** $f(x) = \dfrac{1}{x + 2}$

In Exercises 43–46, describe a basic graph and a sequence of transformations that can be used to produce a graph of the given function.

43. $y = 2(x - 3)^2 - 4$ **44.** $y = -3\sqrt{x + 1}$

45. $y = (3x)^2 - 4$ **46.** $y = -2|x + 4| + 1$

In Exercises 47–50, a graph G is obtained from a graph of y by the sequence of transformations indicated. Write an equation whose graph is G.

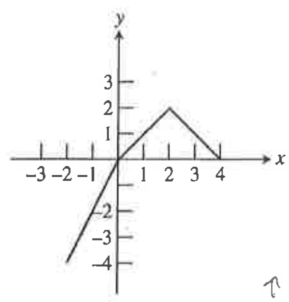
47. $y = x^2$: a vertical stretch by a factor of 3, then a shift right 4 units.

48. $y = x^2$: a shift right 4 units, then a vertical stretch by a factor of 3.

49. $y = |x|$: a shift left 2 units, then a vertical stretch by a factor of 2, and finally a shift down 4 units.

50. $y = |x|$: a shift left 2 units, then a horizontal shrink by a factor of 1/2, and finally a shift down 4 units.

Exercises 51–54 refer to the function f whose graph is shown below.



51. Sketch the graph of $y = 2 + 3f(x + 1)$.

52. Sketch the graph of $y = -f(x + 1) + 1$.

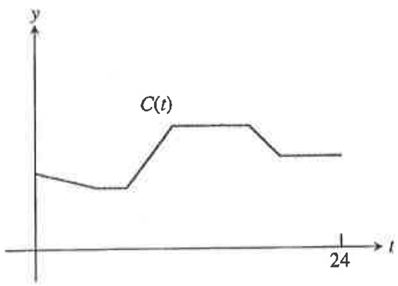
53. Sketch the graph of $y = f(2x)$.

54. Sketch the graph of $y = 2f(x - 1) + 2$.

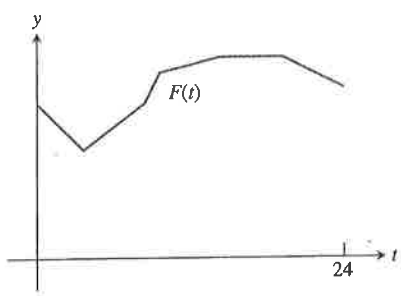
55. Writing to Learn Graph some examples to convince yourself that a reflection and a translation can have a different effect when combined in one order than when combined in the opposite order. Then explain in your own words why this can happen.

56. Writing to Learn Graph some examples to convince yourself that vertical stretches and shrinks do not affect a graph's x-intercepts. Then explain in your own words why this is so.

57. **Celsius vs. Fahrenheit** The graph shows the temperature in degrees Celsius in Windsor, Ontario, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Fahrenheit. [*Hint*: $F(t) = (9/5)C(t) + 32$.]



58. **Fahrenheit vs. Celsius** The graph shows the temperature in degrees Fahrenheit in Mt. Clemens, Michigan, for one 24-hour period. Describe the transformations that convert this graph to one showing degrees Celsius. [*Hint*: $F(t) = (9/5)C(t) + 32$.]



Standardized Test Questions

59. **True or False** The function $y = f(x + 3)$ represents a translation to the right by 3 units of the graph of $y = f(x)$. Justify your answer.

60. **True or False** The function $y = f(x) - 4$ represents a translation down 4 units of the graph of $y = f(x)$. Justify your answer.

In Exercises 61–64, you may use a graphing calculator to answer the question.

61. **Multiple Choice** Given a function f, which of the following represents a vertical stretch by a factor of 3?
(a) $y = f(3x)$ **(b)** $y = f(x/3)$
(c) $y = 3f(x)$ **(d)** $y = f(x)/3$
(e) $y = f(x) + 3$

62. **Multiple Choice** Given a function f, which of the following represents a horizontal translation of 4 units to the right?
(a) $y = f(x) + 4$ **(b)** $y = f(x) - 4$
(c) $y = f(x + 4)$ **(d)** $y = f(x - 4)$
(e) $y = 4f(x)$

63. **Multiple Choice** Given a function f, which of the following represents a vertical translation of 2 units upward, followed by a reflection across the y-axis?
(a) $y = f(-x) + 2$ **(b)** $y = 2 - f(x)$
(c) $y = f(2 - x)$ **(d)** $y = -f(x - 2)$
(e) $y = f(x) - 2$

64. **Multiple Choice** Given a function f, which of the following represents reflection across the x-axis, followed by a horizontal shrink by a factor of 1/2?
(a) $y = -2f(x)$ **(b)** $y = -f(x)/2$
(c) $y = f(-2x)$ **(d)** $y = -f(x/2)$
(e) $y = -f(2x)$

Explorations

65. **International Investment** Table 1.11 shows the price of a share of stock in Debeers Consolidated Mines for the first 8 months of 1999:

Table 1.11 Debeers Consolidated Mines

Month	Price ($)
1	14.25
2	14.89
3	18.94
4	24.50
5	21.44
6	23.88
7	24.75
8	27.19

Source: Salomon Smith Barney.

(a) Graph price (y) as a function of month (x) as a line graph, connecting the points to make a continuous graph.

(b) Explain what transformation you would apply to this graph to produce a graph showing the price of the stock in Japanese yen.

66. **Group Activity** Get with a friend and graph the function $y = x^2$ on both your graphers. Apply a horizontal or vertical stretch or shrink to the function on one of the graphers. Then change the *window* of that grapher to make the two graphs look the same. Can you formulate a general rule for how to find the window?

Extending the Ideas

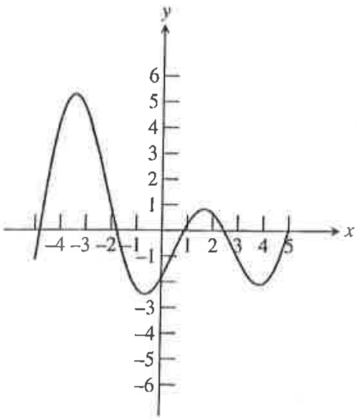
67. The Absolute Value Transformation Graph the function $f(x) = x^4 - 5x^3 + 4x^2 + 3x + 2$ in the viewing window $[-5, 5]$ by $[-10, 10]$. (Put the equation in Y1.)

(a) Study the graph and try to predict what the graph of $y = |f(x)|$ will look like. Then turn Y1 off and graph Y2 = abs (Y1). Did you predict correctly?

(b) Study the original graph again and try to predict what the graph of $y = f(|x|)$ will look like. Then turn Y1 off and graph Y2 = Y1(abs (X)). Did you predict correctly?

(c) Given the graph of $y = g(x)$ shown below, sketch a graph of $y = |g(x)|$.

(d) Given the graph of $y = g(x)$ shown below, sketch a graph of $y = g(|x|)$.



68. Parametric Circles and Ellipses Set your grapher to parametric and radian mode and your window as follows:

Tmin = 0, Tmax = 7, Tstep = 0.1

Xmin = −4.7, Xmax = 4.7, Xscl = 1

Ymin = −3.1, Ymax = 3.1, Yscl = 1

(a) Graph the parametric equations $x = \cos t$ and $y = \sin t$. You should get a circle of radius 1.

(b) Use a transformation of the parametric function of x to produce the graph of an ellipse that is 4 units wide and 2 units tall.

(c) Use a transformation of both parametric functions to produce a circle of radius 3.

(d) Use a transformation of both functions to produce an ellipse that is 8 units wide and 4 units tall.

(You will learn more about ellipses in Chapter 8.)

1.6 MODELING WITH FUNCTIONS

What you'll learn about

- Functions from Formulas
- Functions from Graphs
- Functions from Verbal Descriptions
- Functions from Data

. . . and why

Using a function to model a variable under observation in terms of another variable often allows one to make predictions in practical situations, such as predicting the future growth of a business based on known data.

Functions from Formulas

Now that you have learned more about what functions are and how they behave, we want to return to the modeling theme of Section 1.1. In that section we stressed that one of the goals of this course was to become adept at using numerical, algebraic, and graphical models of the real world in order to solve problems. We now want to focus your attention more precisely on modeling with *functions.*

You have already seen quite a few formulas in the course of your education. Formulas involving two variable quantities always relate those variables implicitly, and quite often the formulas can be solved to give one variable explicitly as a function of the other. In this book we will use a variety of formulas to pose and solve problems algebraically, although we will not assume prior familiarity with those formulas that we borrow from other subject areas (like physics or economics). We *will* assume familiarity with certain key formulas from mathematics.

CHAPTER 1 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–10, match the graph with the corresponding function (a)–(j) from the list below. Use your knowledge of function behavior, *not* your grapher.

(a) $f(x) = x^2 - 1$ **(b)** $f(x) = x^2 + 1$

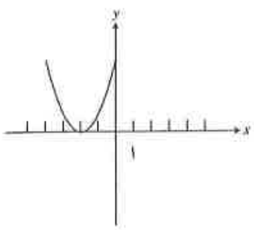
(c) $f(x) = (x - 2)^2$ **(d)** $f(x) = (x + 2)^2$

(e) $f(x) = \frac{x - 1}{2}$ **(f)** $f(x) = |x - 2|$

(g) $f(x) = |x + 2|$ **(h)** $f(x) = -\sin x$

(i) $f(x) = e^x - 1$ **(j)** $f(x) = 1 + \cos x$

1.



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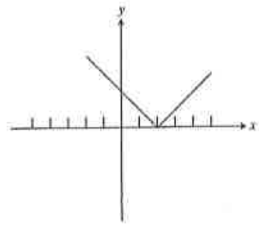

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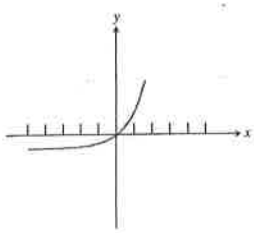

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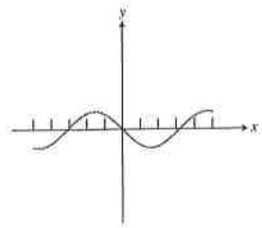

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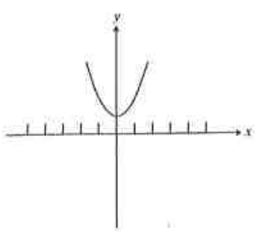

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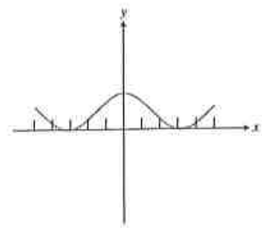

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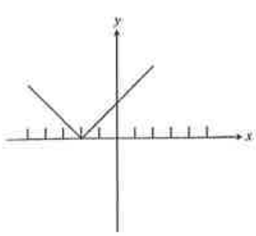

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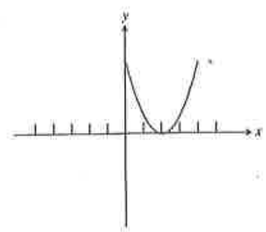

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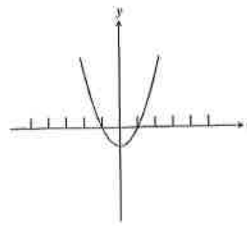
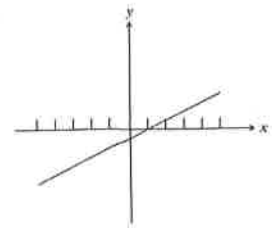


10.



In Exercises 11–18, find **(a)** the domain and **(b)** the range of the function.

11. $g(x) = x^3$ **12.** $f(x) = 35x - 602$

13. $g(x) = x^2 + 2x + 1$ **14.** $h(x) = (x - 2)^2 + 5$

15. $g(x) = 3|x| + 8$ **16.** $k(x) = \sqrt{4 - x^2} - 2$

17. $f(x) = \frac{x}{x^2 - 2x}$ **18.** $k(x) = \frac{1}{\sqrt{9 - x^2}}$

In Exercises 19 and 20, graph the function, and state whether the function is continuous at $x = 0$. If it is discontinuous, state whether the discontinuity is removable or nonremovable.

19. $f(x) = \frac{x^2 - 3}{x + 2}$ **20.** $k(x) = \begin{cases} 2x + 3 & \text{if } x > 0 \\ 3 - x^2 & \text{if } x \le 0 \end{cases}$

In Exercises 21–24, find all **(a)** vertical asymptotes and **(b)** horizontal asymptotes of the graph of the function. Be sure to state your answers as equations of lines.

21. $y = \frac{5}{x^2 - 5x}$ **22.** $y = \frac{3x}{x - 4}$

23. $y = \frac{7x}{\sqrt{x^2 + 10}}$ **24.** $y = \frac{|x|}{x + 1}$

In Exercises 25–28, graph the function and state the intervals on which the function is *increasing*.

25. $y = \frac{x^3}{6}$ **26.** $y = 2 + |x - 1|$

27. $y = \frac{x}{1 - x^2}$ **28.** $y = \frac{x^2 - 1}{x^2 - 4}$

In Exercises 29–32, graph the function and tell whether the function is bounded above, bounded below, or bounded.

29. $f(x) = x + \sin x$ **30.** $g(x) = \frac{6x}{x^2 + 1}$

31. $h(x) = 5 - e^x$ **32.** $k(x) = 1000 + \frac{x}{1000}$

In Exercises 33–36, use a grapher to find all **(a)** relative maximum values and **(b)** relative minimum values of the function. Also state the value of x at which each relative extremum occurs.

33. $y = (x + 1)^2 - 7$ **34.** $y = x^3 - 3x$

35. $y = \frac{x^2 + 4}{x^2 - 4}$ **36.** $y = \frac{4x}{x^2 + 4}$

In Exercises 37–40, graph the function and state whether the function is odd, even, or neither.

37. $y = 3x^2 - 4|x|$

38. $y = \sin x - x^3$

39. $y = \dfrac{x}{e^x}$

40. $y = x \cos(x)$

In Exercises 41–44, find a formula for $f^{-1}(x)$.

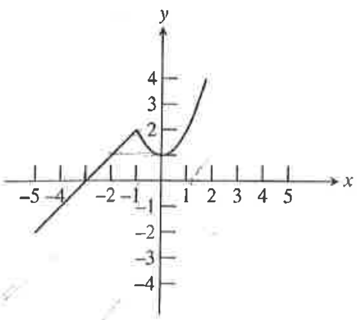
41. $f(x) = 2x + 3$

42. $f(x) = \sqrt[3]{x - 8}$

43. $f(x) = \dfrac{2}{x}$

44. $f(x) = \dfrac{6}{x + 4}$

Exercises 45–52 refer to the function $y = f(x)$ whose graph is given below.



45. Sketch the graph of $y = f(x) - 1$.

46. Sketch the graph of $y = f(x - 1)$.

47. Sketch the graph of $y = f(-x)$.

48. Sketch the graph of $y = -f(x)$.

49. Sketch a graph of the inverse relation.

50. Does the inverse relation define y as a function of x?

51. Sketch a graph of $y = f(|x|)$.

52. Define f algebraically as a piecewise function. [*Hint*: the pieces are translations of two of our "basic" functions.]

In Exercises 53–58, let $f(x) = \sqrt{x}$ and let $g(x) = x^2 - 4$.

53. Find an expression for $(f \circ g)(x)$ and give its domain.

54. Find an expression for $(g \circ f)(x)$ and give its domain.

55. Find an expression for $(fg)(x)$ and give its domain.

56. Find an expression for $\left(\dfrac{f}{g}\right)(x)$ and give its domain.

57. Describe the end behavior of the graph of $y = f(x)$.

58. Describe the end behavior of the graph of $y = f(g(x))$.

In Exercises 59–64, write the specified quantity as a function of the specified variable. Remember that drawing a picture will help.

59. Square inscribed in a circle A square of side s is inscribed in a circle. Write the area of the circle as a function of s.

60. Circle inscribed in a square A circle is inscribed in a square of side s. Write the area of the circle as a function of s.

61. Volume of a cylindrical tank A cylindrical tank with diameter 20 feet is partially filled with oil to a depth of h feet. Write the volume of oil in the tank as a function of h.

62. Draining a cylindrical tank A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining at a constant rate of 2 cubic feet per second. Write the volume of the oil remaining in the tank t seconds later as a function of t.

63. Draining a cylindrical tank A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining at a constant rate of 2 cubic feet per second. Write the depth of the oil remaining in the tank t seconds later as a function of t.

64. Draining a cylindrical tank A cylindrical tank with diameter 20 feet is filled with oil to a depth of 40 feet. The oil begins draining so that the depth of oil in the tank decreases at a constant rate of 2 feet per hour. Write the volume of oil remaining in the tank t hours later as a function of t.

65. The number of new packaged-goods products introduced into the marketplace each year from 1986 to 1997 is shown in Table 1.15:

TABLE 1.15 NEW PACKAGED-GOODS PRODUCTS

Year	New Products
1986	12;436
1987	14,254
1988	13,421
1989	13,382
1990	15,879
1991	15,401
1992	15,886
1993	17,363
1994	21,986
1995	20,808
1996	24,486
1997	25,261

Source: Marketing Intelligence, Ltd.

(a) Sketch a scatter plot of new products (y) as a function of years since 1980 (x). (The values of x will run from 6 to 17.)

(b) Find the equation of the linear regression line.

(c) Based on the regression line, approximately how many new products would be introduced in the year 2000?

66. The winning times in the Women's 100-Meter Freestyle event at the Summer Olympic Games since 1948 are shown in Table 1.16:

TABLE 1.16 WOMEN'S 100-METER FREESTYLE

Year	Time	Year	Time
1948	66.3	1976	55.65
1952	66.8	1980	54.79
1956	62.0	1984	55.92
1960	61.2	1988	54.93
1964	59.5	1992	54.64
1968	60.0	1996	54.5
1972	58.59		

Source: The World Almanac.

(a) Sketch a scatter plot of the times (y) as a function of the years (x) beyond 1900. (The values of x will run from 48 to 96.)

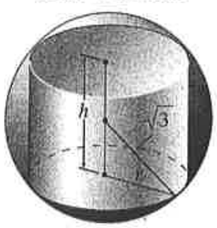
(b) Explain why a linear model cannot be appropriate for these times over the long term.

(c) The points appear to be approaching a horizontal asymptote of $y = 52$. What would this mean about the times in this Olympic event?

(d) Subtract 52 from all of the times so that they will approach an asymptote of $y = 0$. Redo the scatter plot with the new y-values. Now find the exponential regression curve and superimpose its graph on the vertically-shifted scatter plot.

(e) According to the regression curve, what will be the winning time in the Women's 100-Meter Freestyle at the 2000 Summer Olympics?

67. Inscribing a cylinder inside a sphere A right circular cylinder of radius r and height h is inscribed inside a sphere of radius $\sqrt{3}$ inches.



(a) Use the Pythagorean Theorem to write h as a function of r.

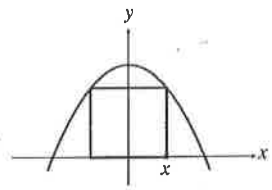
(b) Write the volume V of the cylinder as a function of r.

(c) What values of r are in the domain of V?

(d) Sketch a graph of $V(r)$ over the domain $[0, \sqrt{3}]$.

(e) Use your grapher to find the maximum volume that such a cylinder can have.

68. Inscribing a rectangle under a parabola A rectangle is inscribed between the x-axis and the parabola $y = 36 - x^2$ with one side along the x-axis, as shown in the figure below.



(a) Let x denote the x-coordinate of the point highlighted in the figure. Write the area A of the rectangle as a function of x.

(b) What values of x are in the domain of A?

(c) Sketch a graph of $A(x)$ over the domain.

(d) Use your grapher to find the maximum area that such a rectangle can have.