CHAPTER 6 Project

Parametrizing Ellipses

As you discovered in the Chapter 4 Data Project, it is possible to model the displacement of a swinging pendulum using a sinusoidal equation of the form

$$x = a \sin (b(t - c)) + d$$

where x represents the pendulum's distance from a fixed point and t represents total elapsed time. In fact, a pendulum's velocity behaves sinusoidally as well: $y = ab \cos (b(t - c))$, where y represents the pendulum's velocity and a, b, and c are constants common to both the displacement and velocity equations.

Use a motion detection device to collect distance, velocity, and time data for a pendulum, then determine how a resulting plot of velocity versus displacement (called a phase-space plot) can be modeled using parametric equations.

COLLECTING THE DATA

Construct a simple pendulum by fastening about 1 meter of string to the end of a ball. Collect time, distance, and velocity readings for between 2 and 4 seconds (enough time to capture at least one complete swing of the pendulum). Start the pendulum swinging in front of the detector, then activate the system. The data table below shows a sample set of data collected as a pendulum swung back and forth in front of a CBR where t is total elapsed time in seconds, d = distance from the CBR in meters, v = velocity in meters/second.

t	d	v	t	d	v	t	d	v
0	1.021	0.325	0.7	0.621	−0.869	1.4	0.687	0.966
0.1	1.038	0.013	0.8	0.544	−0.654	1.5	0.785	1.013
0.2	1.023	−0.309	0.9	0.493	−0.359	1.6	0.880	0.826
0.3	0.977	−0.598	1.0	0.473	−0.044	1.7	0.954	0.678
0.4	0.903	−0.819	1.1	0.484	0.263	1.8	1.008	0.378
0.5	0.815	−0.996	1.2	0.526	0.573	1.9	1.030	0.049
0.6	0.715	−0.979	1.3	0.596	0.822	2.0	1.020	−0.260

EXPLORATIONS

1. Create a scatter plot for the data you collected or the data above.
2. With your calculator/computer in function mode, find values for a, b, c, and d so that the equation $y = a \sin (b(x - c)) + d$ (where y is distance and x is time) fits the distance versus time data plot.
3. Make a scatter plot of velocity versus time. Using the same a, b, and c values you found in (2), verify that the equation $y = ab \cos (b(x - c))$ (where y is velocity and x is time) fits the velocity versus time data plot.
4. What do you think a plot of velocity versus distance (with velocity on the vertical axis and distance on the horizontal axis) would look like? Make a rough sketch of your prediction, then create a scatter plot of velocity versus distance. How well did your predicted graph match the actual data plot?
5. With your calculator/computer in parametric mode, graph the parametric curve $x = a \sin (b(t - c)) + d$, $y = ab \cos (b(t - c))$, $0 \le t \le 2$ where x represents distance, y represents velocity, and t is the time parameter. How well does this curve match the scatter plot of velocity versus time?

CHAPTER 7

Systems and Matrices

Scientists studying hemoglobin molecules, as represented in the photo, can make new discoveries by viewing the image on a computer. To see all the details, they may need to move the image up or down (translation), turn it around (rotation), or change the size (scaling). In computer graphics these operations are performed using matrix operations. See a related problem involving scaling a triangle on page 587.

Chapter 7 Overview

Many applications of mathematics in science, engineering, business, and other areas involve the use of systems of equations or inequalities in two or more variables as models for the corresponding problems. We investigate several techniques commonly used to solve such systems; and we investigate matrices, which play a central role in several of these techniques. The information age has made the use of matrices widespread because of their use in handling vast amounts of data.

We decompose a rational function into a sum of simpler rational functions using the method of partial fractions. This technique can be used to analyze a rational function, and is used in calculus to integrate rational functions analytically. Finally, we introduce linear programming, a method used to solve problems concerned with decision making in management science.

7.1 SOLVING SYSTEMS OF TWO EQUATIONS

What you'll learn about

- The Method of Substitution
- Solving Systems Graphically
- The Method of Elimination
- Applications

... and why

Many applications in business and science can be modeled using systems of equations.

The Method of Substitution

Here is an example of a system of two linear equations in the two variables x and y:

$$2x - y = 10$$
$$3x + 2y = 1.$$

A **solution of a system** of two equations in two variables is an ordered pair of real numbers that is a solution of each equation. For example, the ordered pair $(3, -4)$ is a solution to the above system. We can verify this by showing that $(3, -4)$ is a solution of each equation. Substituting $x = 3$ and $y = -4$ into each equation, we obtain

$$2x - y = 2(3) - (-4) = 6 + 4 = 10,$$
$$3x + 2y = 3(3) + 2(-4) = 9 - 8 = 1.$$

So, both equations are satisfied.

We have **solved the system of equations** when we have found all its solutions. In Example 1, we use the method of substitution to see that $(3, -4)$ is the only solution of this system.

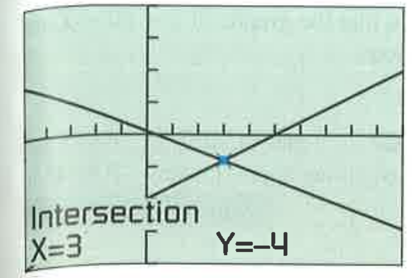
EXAMPLE 1 Using the substitution method

Solve the system

$$2x - y = 10$$
$$3x + 2y = 1.$$

SOLUTION

Solve Algebraically Solving the first equation for y yields $y = 2x - 10$. Then substitute the expression for y into the second equation.



[–5, 10] by [–20, 20]

FIGURE 7.1 The two lines $y = 2x - 10$ and $y = -1.5x + 0.5$ intersect in the point $(3, -4)$. (Example 1)

$3x + 2y = 1$	Second equation
$3x + 2(2x - 10) = 1$	Replace y by $2x - 10$.
$3x + 4x - 20 = 1$	Distributive property
$7x = 21$	Collect like terms.
$x = 3$	Divide by 7.
$y = -4$	Use $y = 2x - 10$.

Support Graphically

The graph of each equation is a line. Figure 7.1 shows that the two lines intersect in the single point $(3, -4)$.

Interpret

The solution of the system is $x = 3$, $y = -4$, or the ordered pair $(3, -4)$.

Now try Exercise 5.

The method of substitution can sometimes be applied when the equations in the system are not linear, as illustrated in Example 2.

EXAMPLE 2 Solving a nonlinear system by substitution

Find the dimensions of a rectangular garden that has perimeter 100 ft and area 300 ft^2.

SOLUTION

Model

Let x and y be the lengths of adjacent sides of the garden (Figure 7.2). Then

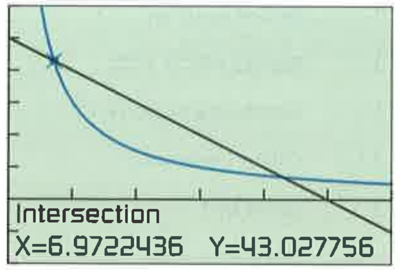
$2x + 2y = 100$	Perimeter is 100.
$xy = 300.$	Area is 300.

Solve Algebraically

Solving the first equation for y yields $y = 50 - x$. Then substitute the expression for y into the second equation.

$xy = 300$	Second equation
$x(50 - x) = 300$	Replace y by $50 - x$.
$50x - x^2 = 300$	Distributive property
$x^2 - 50x + 300 = 0$	
$x = \dfrac{50 \pm \sqrt{(-50)^2 - 4(300)}}{2}$	Quadratic formula
$x = 6.972\ldots$ or $x = 43.027\ldots$	Evaluate.
$y = 43.027\ldots$ or $y = 6.972\ldots$	Use $y = 50 - x$.

FIGURE 7.2 The rectangular garden in Example 2.

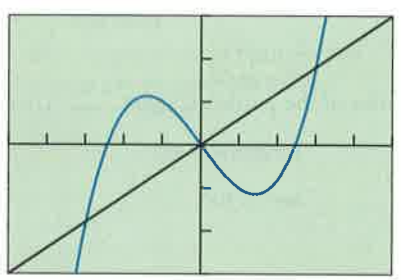


[0, 60] by [−20, 60]

FIGURE 7.3 We can assume $x \geq 0$ and $y \geq 0$ because x and y are lengths. (Example 2)

ROUNDING AT THE END

In Example 2, we did *not* round the values found for x until we computed the values for y. For the sake of accuracy, do not round *intermediate results*. Carry all decimals on your calculator computations and then round the final answer(s).



[−5, 5] by [−15, 15]

FIGURE 7.4 The graphs of $y = x^3 - 6x$ and $y = 3x$ have three points of intersection. (Example 3)

Support Graphically Figure 7.3 shows that the graphs of $y = 50 - x$ and $y = 300/x$ have two points of intersections.

Interpret

The two ordered pairs (6.972 . . . , 43.027 . . .) and (43.027 . . . , 6.972 . . .) produce the same rectangle whose dimensions are approximately 7 ft by 43 ft.

Now try Exercise 11.

EXAMPLE 3 Solving a nonlinear system algebraically

Solve the system

$$y = x^3 - 6x$$
$$y = 3x.$$

Support your solution graphically.

SOLUTION

Substituting the value of y from the first equation into the second equation yields

$$x^3 - 6x = 3x$$
$$x^3 - 9x = 0$$
$$x(x - 3)(x + 3) = 0$$
$$x = 0, x = 3, x = -3 \qquad \text{Zero factor property}$$
$$y = 0, y = 9, y = -9 \qquad \text{Use } y = 3x.$$

The system of equations has three solutions: $(-3, -9)$, $(0, 0)$, and $(3, 9)$.

Support Graphically The graphs of the two equations in Figure 7.4 suggests that the three solutions found algebraically are correct.

Now try Exercise 13.

Solving Systems Graphically

Sometimes the method of substitution leads to an equation in one variable that we are not able to solve using the standard algebraic techniques we have studied in this text. In these cases we can solve the system graphically by finding intersections as illustrated in Exploration 1.

EXPLORATION 1 Solving a System Graphically

Consider the system:

$$y = \ln x$$
$$y = x^2 - 4x + 2$$

1. Draw the graphs of the two equations in the $[0, 10]$ by $[-5, 5]$ viewing window.
2. Use the graph in part 1 to find the coordinates of the points of intersection shown in the viewing window.
3. Use your knowledge about the graphs of logarithmic and quadratic functions to explain why this system has exactly two solutions.

Substituting the expression for y of the first equation of Exploration 1 into the second equation yields

$$\ln x = x^2 - 4x + 2.$$

We have no standard algebraic technique to solve this equation.

The Method of Elimination

Consider a system of two linear equations in x and y. To **solve by elimination**, we rewrite the two equations as two equivalent equations so that one of the variables has opposite coefficients. Then we add the two equations to eliminate that variable.

EXAMPLE 4 Using the elimination method

Solve the system

$$2x + 3y = 5$$
$$-3x + 5y = 21.$$

SOLUTION

Solve Algebraically Multiply the first equation by 3 and the second equation by 2 to obtain

$$6x + 9y = 15$$
$$-6x + 10y = 42.$$

Then add the two equations to eliminate the variable x.

$$19y = 57$$

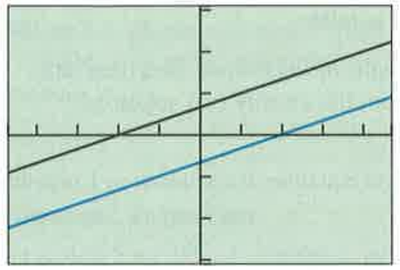
Next divide by 19 to solve for y.

$$y = 3$$

Finally, substitute $y = 3$ into either of the two original equations to determine that $x = -2$.

The solution of the original system is $(-2, 3)$.

Now try Exercise 19.



[–4.7, 4.7] by [–3.1, 3.1]

FIGURE 7.5 The graph of the two lines in Example 5 in this square viewing window appear to be parallel.

EXAMPLE 5 Finding no solution

Solve the system

$$x - 3y = -2$$
$$2x - 6y = 4.$$

SOLUTION We use the elimination method.

Solve Algebraically

$$-2x + 6y = 4 \quad \text{Multiply first equation by } -2.$$
$$2x - 6y = 4 \quad \text{Second equation}$$
$$0 = 8 \quad \text{Add.}$$

The last equation is true for *no* values of x and y. The system has no solution.

Support Graphically

Figure 7.5 suggests that the two lines that are the graphs of the two equations in the system are parallel. Solving for y in each equation yields

$$y = \frac{1}{3}x + \frac{2}{3}$$
$$y = \frac{1}{3}x - \frac{2}{3}.$$

The two lines have the same slope of 1/3 and are therefore parallel.

Now try Exercise 23.

An easy way to determine the *number of solutions* of a system of two linear equations in two variables is to look at the graphs of the two lines. There are three possibilities. The two lines can intersect in a single point, producing exactly *one* solution as in Examples 1 and 4. The two lines can be parallel, producing *no* solution as in Example 5. The two lines can be the same, producing infinitely many solutions as illustrated in Example 6.

EXAMPLE 6 Finding infinitely many solutions

Solve the system

$$4x - 5y = 2$$
$$-12x + 15y = -6.$$

SOLUTION

$$12x - 15y = 6 \quad \text{Multiply first equation by 3.}$$
$$-12x + 15y = -6 \quad \text{Second equation}$$
$$0 = 0 \quad \text{Add.}$$

The last equation is true for all values of x and y. Thus, every ordered pair that satisfies one equation satisfies the other equation. The system has infinitely many solutions.

Another way to see that there are infinitely many solutions is to solve each equation for y. Both equations yield

$$y = \frac{4}{5}x - \frac{2}{5}.$$

The two lines are the same.

Now try Exercise 25.

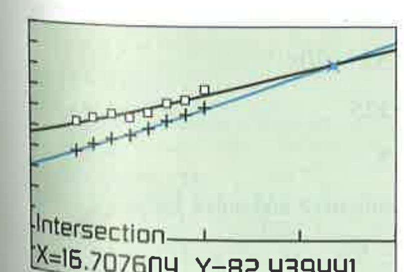
Applications

Table 7.1 shows the personal consumption expenditures (in billions of dollars) for dentists and health insurance in the U.S. for several years.

TABLE 7.1 U.S. PERSONAL CONSUMPTION EXPENDITURES

Year	Dentists (billions of dollars)	Health Insurance (billions of dollars)
1993	53.5	39.9
1994	55.8	42.9
1995	58.0	46.5
1996	56.6	48.4
1997	59.3	51.9
1998	63.6	55.1
1999	65.3	58.3
2000	70.0	62.1

Source: Bureau of Economic Analysis, U.S. Department of Commerce, as reported in The World Almanac and Book of Facts, 2002.



[0, 20] by [–20, 100]

FIGURE 7.6 The scatter plot and regression equations for the data in Table 7.1. Dentist (□), health insurance (+). (Example 7)

EXAMPLE 7 Estimating personal expenditures with linear models

(a) Find linear regression equations for the U.S. personal consumption expenditures for dentists and health insurance in Table 7.1. Superimpose their graphs on a scatter plot of the data.

(b) Use the models in (a) to estimate when the U.S. personal consumption expenditures for dentists will be the same as that for health insurance and the corresponding amount.

SOLUTION

(a) Let $x = 0$ stand for 1990, $x = 1$ for 1991, and so forth. We use a graphing calculator to find linear regression equations for the amount of expenditures for dentists, y_D, and the amount of expenditures for health insurance, y_{HI}:

$$y_D \approx 2.1726\,x + 46.1405$$
$$y_{HI} \approx 3.1155\,x + 30.3869$$

Figure 7.6 shows the two regression equations together with a scatter plot of the two sets of data.

(b) Figure 7.6 shows that the graphs of y_D and y_{HI} intersect at approximately (16.71, 82.44). $x = 16$ stands for 2006, so Figure 7.6 suggests that the personal consumption expenditures for dentists and for health insurance will both be about 82.4 billion sometime during 2006.

Now try Exercise 45.

Suppliers will usually increase production, x, if they can get higher prices, p, for their products. So, as one variable increases, the other also increases. Normal mathematical practice would be to use p as the independent variable and x as the dependent variable. However, most economists put x on the horizontal axis and p on the vertical axis. In keeping with this practice, we write $p = f(x)$ for a **supply curve**. On one hand, as the price increases (vertical axis) so does the willingness for suppliers to increase production x (horizontal axis).

On the other hand, the demand, x, for a product by consumers will decrease as the price, p, goes up. So, as one variable increases, the other decreases. Again economists put x (demand) on the horizontal axis and p (price) on the vertical axis, even though it seems like p should be the dependent variable. In keeping with this practice, we write $p = g(x)$ for a **demand curve**.

Finally, a point where the supply curve and demand curve intersect is an **equilibrium point**. The corresponding price is the **equilibrium price**.

EXAMPLE 8 Determining the equilibrium price

Nibok Manufacturing has determined that production and price of a new tennis shoe should be geared to the equilibrium point for this system of equations.

$$p = 160 - 5x \quad \text{Demand curve}$$
$$p = 35 + 20x \quad \text{Supply curve}$$

The price, p, is in dollars and the number of shoes, x, is in millions of pairs. Find the equilibrium point.

SOLUTION We use substitution to solve the system.

$$160 - 5x = 35 + 20x$$
$$25x = 125$$
$$x = 5$$

Substitute this value of x into the demand curve and solve for p.

$$p = 160 - 5x$$
$$p = 160 - 5(5) = 135$$

The equilibrium point is (5, 135). The equilibrium price is \$135, the price for which supply and demand will be equal at 5 million pairs of tennis shoes.

Now try Exercise 43.

QUICK REVIEW 7.1 (For help, go to Sections P.4 and P.5.)

In Exercises 1 and 2, solve for y in terms of x.

1. $2x + 3y = 5$
2. $xy + x = 4$

In Exercises 3–6, solve the equation algebraically.

3. $3x^2 - x - 2 = 0$
4. $2x^2 + 5x - 10 = 0$
5. $x^3 = 4x$
6. $x^3 + x^2 = 6x$
7. Write an equation for the line through the point $(-1, 2)$ and parallel to the line $4x + 5y = 2$.
8. Write an equation for the line through the point $(-1, 2)$ and perpendicular to the line $4x + 5y = 2$.
9. Write an equation equivalent to $2x + 3y = 5$ with coefficient of x equal to -4.
10. Find the points of intersection of the graphs of $y = 3x$ and $y = x^3 - 6x$ graphically.

SECTION 7.1 EXERCISES

In Exercises 1 and 2, determine whether the ordered pair is a solution of the system.

1. $5x - 2y = 8$, $2x - 3y = 1$
   (a) $(0, 4)$ (b) $(2, 1)$ (c) $(-2, -9)$
2. $y = x^2 - 6x + 5$, $y = 2x - 7$
   (a) $(2, -3)$ (b) $(1, -5)$ (c) $(6, 5)$

In Exercises 3–12, solve the system by substitution.

3. $x + 2y = 5$, $y = -2$
4. $x = 3$, $x - y = 20$
5. $3x + y = 20$, $x - 2y = 10$
6. $2x - 3y = -23$, $x + y = 0$
7. $2x - 3y = -7$, $4x + 5y = 8$
8. $3x + 2y = -5$, $2x - 5y = -16$
9. $x - 3y = 6$, $-2x + 6y = 4$
10. $3x - y = -2$, $-9x + 3y = 6$
11. $y = x^2$, $y - 9 = 0$
12. $x = y + 3$, $x - y^2 = 3y$

In Exercises 13–18, solve the system algebraically. Support your answer graphically.

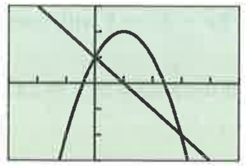
13. $y = 6x^2$, $7x + y = 3$
14. $y = 2x^2 + x$, $2x + y = 20$
15. $y = x^3 - x^2$, $y = 2x^2$
16. $y = x^3 + x^2$, $y = -x^2$
17. $x^2 + y^2 = 9$, $x - 3y = -1$
18. $x^2 + y^2 = 16$, $4x + 7y = 13$

In Exercises 19–26, solve the system by elimination.

19. $x - y = 10$, $x + y = 6$
20. $2x + y = 10$, $x - 2y = -5$
21. $3x - 2y = 8$, $5x + 4y = 28$
22. $4x - 5y = -23$, $3x + 4y = 6$
23. $2x - 4y = -10$, $-3x + 6y = -21$
24. $2x - 4y = 8$, $-x + 2y = -4$
25. $2x - 3y = 5$, $-6x + 9y = -15$
26. $2x - y = 3$, $-4x + 2y = 5$

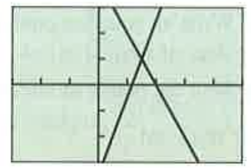
In Exercises 27–30, use the graph to estimate any solutions of the system.

27. $y = 1 + 2x - x^2$
$y = 1 - x$



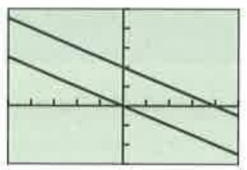
$[-3, 5]$ by $[-3, 3]$

28. $6x - 2y = 7$
$2x + y = 4$



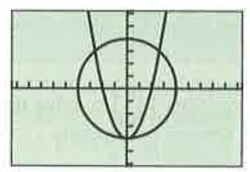
$[-3, 5]$ by $[-3, 3]$

29. $x + 2y = 0$
$0.5x + y = 2$



$[-5, 5]$ by $[-3, 5]$

30. $x^2 + y^2 = 16$
$y + 4 = x^2$



$[-9.4, 9.4]$ by $[-6.2, 6.2]$

In Exercises 31–34, use graphs to determine the number of solutions the system has.

31. $3x + 5y = 7$
$4x - 2y = -3$

32. $3x - 9y = 6$
$2x - 6y = 1$

33. $2x - 4y = 6$
$3x - 6y = 9$

34. $x - 7y = 9$
$3x + 4y = 1$

In Exercises 35–42, solve the system graphically. Support your answer numerically.

35. $y = \ln x$
$1 = 2x + y$

36. $y = 3 \cos x$
$1 = 2x - y$

37. $y = x^3 - 4x$
$4 = x - 2y$

38. $y = x^2 - 3x - 5$
$1 = 2x - y$

39. $x^2 + y^2 = 4$
$x + 2y = 2$

40. $x^2 + y^2 = 4$
$x - 2y = 2$

41. $x^2 + y^2 = 9$
$y = x^2 - 2$

42. $x^2 + y^2 = 9$
$y = 2 - x^2$

In Exercises 43 and 44, find the equilibrium point for the given demand and supply curve.

43. $p = 200 - 15x$
$p = 50 + 25x$

44. $p = 15 - \dfrac{7}{100}x$
$p = 2 + \dfrac{3}{100}x$

45. Medicare Expenditure Table 7.2 shows expenditures (in billions of dollars) for benefits and administrative cost from federal hospital and medical insurance trust funds for several years. Let $x = 0$ stand for 1980, $x = 1$ for 1981, and so forth.

(a) Find the quadratic regression equation and superimpose its graph on a scatter plot of the data.

(b) Find the logistic regression equation and superimpose its graph on a scatter plot of the data.

(c) When will the two models predict expenditures of 240 billion dollars?

(d) Writing to Learn Explain the long range implications of using the quadratic regression equation to predict future expenditures.

(e) Writing to Learn Explain the long range implications of using the logistic regression equation to predict future expenditures.

Table 7.2 Medicare National Health Expenditures

Year	Expenditures (billions)
1990	110.2
1993	148.3
1994	166.2
1995	184.8
1996	200.3
1997	211.2
1998	211.4
1999	213.6

Source: U.S. Health Care Financing Administration, Health Care Financing Review, Summer 2001, in Statistical Abstract of the U.S., 2001.

46. Personal Income Table 7.3 gives the total personal income (in billions of dollars) for residents of the states of Iowa and Nevada for several years. Let $x = 0$ stand for 1990, $x = 1$ for 1991, and so forth.

(a) Find the linear regression equation for the Iowa data and superimpose its graph on a scatter plot of the Iowa data.

(b) Find the linear regression equation for the Nevada data and superimpose its graph on a scatter plot of the Nevada data.

(c) Using the models in (a) and (b), when will the personal income of the two states be the same?

Table 7.3 Total Personal Income

Year	Iowa (billions)	Nevada (billions)
1990	48.3	25.2
1999	73.5	56.1
2000	78.2	61

Source: U.S. Bureau of Economic Analysis, Survey of Current Business, May 1998, in Statistical Abstract of the U.S., 2001.

47. Population Table 7.4 gives the population (in thousands) of the states of Arizona and Massachusetts for several years. Let $x = 0$ stand for 1980, $x = 1$ for 1981, and so forth.

(a) Find the linear regression equation for Arizona's data and superimpose its graph on a scatter plot of Arizona's data.

(b) Find the linear regression equation for Massachusetts's data and superimpose its graph on a scatter plot of Massachusetts data.

(c) Using the models in (a) and (b), when will the population of the two states be the same?

Table 7.4 Population

Year	Arizona (in thousands)	Massachusetts (in thousands)
1980	2718	5737
1990	3665	6016
1991	3762	5999
1992	3867	5993
1993	3993	6011
1994	4148	6031
1995	4307	6062
1996	4432	6085
1997	4552	6115
1998	4667	6144
1999	4778	6175
2000	5131	6349

Source: U.S. Bureau of the Census, in Statistical Abstract of the U.S., 2001.

48. Group Activity Describe all possibilities for the number of solutions to a system of two equations in two variables if the graphs of the two equations are (a) a line and a circle, and (b) a circle and a parabola.

49. Garden Problem Find the dimensions of a rectangle with a perimeter of 200 m and an area of 500 m^2.

50. Cornfield Dimensions Find the dimensions of a rectangular cornfield with a perimeter of 220 yd and an area of 3000 yd^2.

51. Rowing Speed Hank can row a boat 1 mi upstream (against the current) in 24 min. He can row the same distance downstream in 13 min. If both the rowing speed and current speed are constant, find Hank's rowing speed and the speed of the current.

52. Airplane Speed An airplane flying with the wind from Los Angeles to New York City takes 3.75 hr. Flying against the wind, the airplane takes 4.4 hr for the return trip. If the air distance between Los Angeles and New York is 2500 mi and the airplane speed and wind speed are constant, find the airplane speed and the wind speed.

53. Food Prices At Philip's convenience store the total cost of one medium and one large soda is \$1.74. The large soda costs \$0.16 more than the medium soda. Find the cost of each soda.

54. Nut Mixture A 5-lb nut mixture is worth \$2.80 per pound. The mixture contains peanuts worth \$1.70 per pound and cashews worth \$4.55 per pound. How many pounds of each type of nut are in the mixture?

55. Connecting Algebra and Functions Determine a and b so that the graph of $y = ax + b$ contains the two points $(-1, 4)$ and $(2, 6)$.

56. Connecting Algebra and Functions Determine a and b so that the graph of $ax + by = 8$ contains the two points $(2, -1)$ and $(-4, -6)$.

57. Rental Van Pedro has two plans to choose from to rent a van.

Company A: a flat fee of \$40 plus 10 cents a mile.

Company B: a flat fee of \$25 plus 15 cents a mile.

(a) How many miles can Pedro drive in order to be charged the same amount by the two companies?

(b) Writing to Learn Give reasons why Pedro might choose one plan over the other. Explain.

58. Salary Package Stephanie is offered two different salary options to sell major household appliances.

Plan A: a \$300 weekly salary plus 5% of her sales.

Plan B: a \$600 weekly salary plus 1% of her sales.

(a) What must Stephanie's sales be to earn the same amount on the two plans?

(b) Writing to Learn Give reasons why Stephanie might choose one plan over the other. Explain.

Standardized Test Questions

59. True or False Let a and b be real numbers. The following system of equations can have exactly two solutions:
$2x + 5y = a$
$3x - 4y = b.$
Justify your answer.

60. True or False If the resulting equation after using elimination correctly on a system of two linear equations in two variables is $7 = 0$, then the system has infinitely many solutions. Justify your answer.

In Exercises 61–64, solve the problem without using a calculator.

61. Multiple Choice Which of the following is a solution of the system $2x - 3y = 12$
$x + 2y = -1$?
(a) $(-3, 1)$ **(b)** $(-1, 0)$ **(c)** $(3, -2)$
(d) $(3, 2)$ **(e)** $(6, 0)$

62. Multiple Choice Which of the following cannot be the number of solutions of a system of two equations in two variables whose graphs are a circle and a parabola?
(a) 0 **(b)** 1 **(c)** 2 **(d)** 3 **(e)** 5

63. Multiple Choice Which of the following cannot be the number of solutions of a system of two equations in two variables whose graphs are parabolas?
(a) 1 **(b)** 2 **(c)** 4
(d) 5 **(e)** Infinitely many

64. Multiple Choice Which of the following is the number of solutions of a system of two linear equations in two variables if the resulting equation after using elimination correctly is $4 = 4$?
(a) 0 **(b)** 1 **(c)** 2
(d) 3 **(e)** Infinitely many

Explorations

65. An Ellipse and a Line Consider the system of equations

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$
$$x + y = 1.$$

(a) Solve the equation $x^2/4 + y^2/9 = 1$ for y in terms of x to determine the two implicit functions determined by the equation.
(b) Solve the system of equations graphically.
(c) Use substitution to confirm the solutions found in (b).

66. A Hyperbola and a Line Consider the system of equations

$$\frac{x^2}{4} - \frac{y^2}{9} = 1$$
$$x - y = 0.$$

(a) Solve the equation $x^2/4 - y^2/9 = 1$ for y in terms of x to determine the two implicit functions determined by the equation.
(b) Solve the system of equations graphically.
(c) Use substitution to confirm the solutions found in (b).

Extending the Ideas

In Exercises 67 and 68, use the elimination method to solve the system of equations.

67. $x^2 - 2y = -6$
$x^2 + y = 4$

68. $x^2 + y^2 = 1$
$x^2 - y^2 = 1$

In Exercises 69 and 70, $p(x)$ is the demand curve. The total revenue if x units are sold is $R = px$. Find the number of units sold that gives the maximum revenue.

69. $p = 100 - 4x$

70. $p = 80 - x^2$

7.2 MATRIX ALGEBRA

What you'll learn about
- Matrices
- Matrix Addition and Subtraction
- Matrix Multiplication
- Identity and Inverse Matrices
- Determinant of a Square Matrix
- Applications

. . . and why
Matrix algebra provides a powerful technique to manipulate large data sets and solve the related problems that are modeled by the matrices.

Matrices

A *matrix* is a rectangular array of numbers. Matrices provide an efficient way to solve systems of linear equations and to record data. The tables of data presented in this textbook are examples of matrices.

Definition Matrix

Let m and n be positive integers. An **$m \times n$ matrix** (read "m by n matrix") is a rectangular array of m rows and n columns of real numbers.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We also use the shorthand notation $[a_{ij}]$ for this matrix.

Each **element**, or **entry**, a_{ij}, of the matrix uses *double subscript* notation. The **row subscript** is the first subscript i, and the **column subscript** is j. The element a_{ij} is in the ith row and jth column. In general, the **order of an $m \times n$ matrix** is $m \times n$. If $m = n$, the matrix is a **square matrix**. Two matrices are **equal matrices** if they have the same order and their corresponding elements are equal.

EXAMPLE 1 Determining the order of a matrix

(a) The matrix $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & 4 \end{bmatrix}$ has order 2×3.

(b) The matrix $\begin{bmatrix} 1 & -1 \\ 0 & 4 \\ 2 & -1 \\ 3 & 2 \end{bmatrix}$ has order 4×2.

(c) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ has order 3×3 and is a square matrix.

Now try Exercise 1.

Historical Note
Methods used by the Chinese between 200 BC and 100 BC to solve problems involving several unknowns were similar to modern methods which use matrices. Matrices were formally developed in the 18th century by several mathematicians, including Leibniz, Cauchy, and Gauss.

Matrix Addition and Subtraction

We add or subtract two matrices of the same order by adding or subtracting their corresponding entries. Matrices of different orders can *not* be added or subtracted.

Definition Matrix Addition and Matrix Subtraction

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of order $m \times n$.

1. The **sum $A + B$** is the $m \times n$ matrix

$$A + B = [a_{ij} + b_{ij}].$$

2. The **difference $A - B$** is the $m \times n$ matrix

$$A - B = [a_{ij} - b_{ij}].$$

Power of Matrix Algebra

The result in Example 2 is fairly simple, but it is significant that we found (essentially) 24 pieces of information with a single mathematical operation. That is the power of matrix algebra.

EXAMPLE 2 Using matrix addition

Matrix A gives the mean SAT verbal scores for the six New England states over the time period from 1998 to 2001. (*Source: The College Board, World Almanac and Book of Facts, 2002.*) Matrix B gives the mean SAT mathematics scores for the same 4-year period. Express the mean combined scores for the New England states from 1998 to 2001 as a single matrix.

$A =$

	98	99	00	01
CT	510	510	508	509
ME	504	507	504	506
MA	508	511	511	511
NH	523	520	520	520
RI	501	495	505	500
VT	508	514	513	511

$B =$

	98	99	00	01
CT	509	509	509	510
ME	501	503	500	500
MA	508	511	513	515
NH	520	518	519	516
RI	495	504	500	501
VT	504	506	508	506

SOLUTION The combined scores can be obtained by adding the two matrices:

$A + B =$

	98	99	00	01
CT	1019	1019	1017	1019
ME	1005	1010	1004	1006
MA	1016	1022	1024	1026
NH	1043	1038	1039	1036
RI	996	999	1005	1001
VT	1012	1020	1021	1017

Now try Exercise 11.

When we work with matrices, real numbers are **scalars**. The product of the real number k and the $m \times n$ matrix $A = [a_{ij}]$ is the $m \times n$ matrix

$$kA = [ka_{ij}].$$

The matrix $kA = [ka_{ij}]$ is a **scalar multiple of A**.

EXAMPLE 3 Using scalar multiplication

A consumer advocacy group has computed the mean retail prices for brand name products and generic products at three different stores in a major city. The prices are shown in the 3×2 matrix below.

	Brand	Generic
Store A	3.97	3.64
Store B	3.78	3.69
Store C	3.75	3.67

The city has a combined sales tax of 7.25%. Construct a matrix showing the comparative prices with sales tax included.

SOLUTION Multiply the original matrix by the scalar 1.0725 to add the sales tax to every price.

$$1.0725 \times \begin{bmatrix} 3.97 & 3.64 \\ 3.78 & 3.69 \\ 3.75 & 3.67 \end{bmatrix} \approx \begin{matrix} \text{Store A} \\ \text{Store B} \\ \text{Store C} \end{matrix} \overset{\text{Brand Generic}}{\begin{bmatrix} 4.26 & 3.90 \\ 4.05 & 3.96 \\ 4.02 & 3.94 \end{bmatrix}}$$

Now try Exercise 13.

Matrices inherit many properties possessed by the real numbers. Let $A = [a_{ij}]$ be any $m \times n$ matrix. The $m \times n$ matrix $O = [0]$ consisting entirely of zeros is the **zero matrix** because $A + O = A$. In other words, O is the **additive identity** for the set of all $m \times n$ matrices. The $m \times n$ matrix $B = [-a_{ij}]$ consisting of the *additive inverses* of the entries of A is the **additive inverse of A** because $A + B = O$. We also write $B = -A$. Just as with real numbers,

$$A - B = [a_{ij} - b_{ij}] = [a_{ij} + (-b_{ij})] = [a_{ij}] + [-b_{ij}] = A + (-B).$$

Thus, subtracting B from A is the same as adding the additive inverse of B to A.

EXPLORATION 1 Computing with Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be 2×2 matrices with $a_{ij} = 3i - j$ and $b_{ij} = i^2 + j^2 - 3$ for $i = 1, 2$ and $j = 1, 2$.

1. Determine A and B.
2. Determine the additive inverse $-A$ of A and verify that $A + (-A) = [0]$. What is the order of $[0]$?
3. Determine $3A - 2B$.

Matrix Multiplication

To form the *product AB* of two matrices, the number of columns of the matrix A on the left must be equal to the number of rows of the matrix B on the right. In this case, any row of A has the same number of entries as any column of B. Each entry of the product is obtained by summing the products of the entries of a row of A by the corresponding entries of a column of B.

Definition Matrix Multiplication

Let $A = [a_{ij}]$ be an $m \times r$ matrix and $B = [b_{ij}]$ an $r \times n$ matrix. The **product $AB = [c_{ij}]$** is the $m \times n$ matrix where
$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$.

The key to understanding how to form the product of any two matrices is to first consider the product of a $1 \times r$ matrix $A = [a_{1j}]$ with an $r \times 1$ matrix $B = [b_{j1}]$. According to the definition, $AB = [c_{11}]$ is the 1×1 matrix where $c_{11} = a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1r}b_{r1}$. For example, the product AB of the 1×3 matrix A and the 3×1 matrix B, where

$$A = [1 \quad 2 \quad 3] \quad \text{and} \quad B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

is

$$A \cdot B = [1 \quad 2 \quad 3] \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = [1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6] = [32].$$

Then, the ij-entry of the product AB of an $m \times r$ matrix with an $r \times n$ matrix is the product of the ith row of A, considered as a $1 \times r$ matrix, with the jth column of B, considered as a $r \times 1$ matrix, as illustrated in Example 4.

EXAMPLE 4 Finding the product of two matrices

Find the product AB if possible, where

(a) $A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 1 & -3 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ 2 & 1 \end{bmatrix}$.

SOLUTION

(a) The number of columns of A is 3 and the number of rows of B is 3, so the product AB is defined. The product $AB = [c_{ij}]$ is a 2×2 matrix where

$$c_{11} = [2 \quad 1 \quad -3] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2 \cdot 1 + 1 \cdot 0 + (-3) \cdot 1 = -1,$$

$$c_{12} = [2 \quad 1 \quad -3] \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = 2 \cdot (-4) + 1 \cdot 2 + (-3) \cdot 0 = -6,$$

$$c_{21} = [0 \quad 1 \quad 2] \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 1 = 2,$$

$$c_{22} = [0 \quad 1 \quad 2] \begin{bmatrix} -4 \\ 2 \\ 0 \end{bmatrix} = 0 \cdot (-4) + 1 \cdot 2 + 2 \cdot 0 = 2.$$


```
[A] [B]
        [[-1  -6]
         [2   2 ]]
```

FIGURE 7.7 The matrix product AB of Example 4. Notice that the grapher displays the rows of the product as 1×2 matrices.

Thus, $AB = \begin{bmatrix} -1 & -6 \\ 2 & 2 \end{bmatrix}$. Figure 7.7 supports this computation.

(b) The number of columns of A is 3 and the number of rows of B is 2, so the product AB is *not* defined. Now try Exercise 19.

EXAMPLE 5 Using matrix multiplication

A florist makes three different cut flower arrangements for Mother's Day (I, II, and III), each involving roses, carnations, and lilies. Matrix A shows the number of each type of flower used in each arrangement.

$$A = \begin{matrix} \text{Roses} \\ \text{Carnations} \\ \text{Lilies} \end{matrix} \overset{\begin{matrix} \text{I} & \text{II} & \text{III} \end{matrix}}{\begin{bmatrix} 5 & 8 & 7 \\ 6 & 6 & 7 \\ 4 & 3 & 3 \end{bmatrix}}$$

The florist can buy his flowers from two different wholesalers (W1 and W2), but wants to give all his business to one or the other. The cost of the three flower types from the two wholesalers is shown in matrix B.

$$B = \begin{matrix} \text{Roses} \\ \text{Carnations} \\ \text{Lilies} \end{matrix} \overset{\begin{matrix} \text{W1} & \text{W2} \end{matrix}}{\begin{bmatrix} 1.50 & 1.35 \\ 0.95 & 1.00 \\ 1.30 & 1.35 \end{bmatrix}}$$

Construct a matrix showing the cost of making each of the three flower arrangements from flowers supplied by the two different wholesalers.

SOLUTION We can use the labeling of the matricies to help us. We want the columns of A to match up with the rows of B (since that's how the matrix multiplication works). We therefore switch the rows and columns of A to get the flowers along the columns. (The new matrix is called the **transpose** of A, denoted by A^T.) We then find the product A^TB:

$$\begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} \overset{\text{Rose Carn Lily}}{\begin{bmatrix} 5 & 6 & 4 \\ 8 & 6 & 3 \\ 7 & 7 & 3 \end{bmatrix}} \times \begin{matrix} \text{Rose} \\ \text{Carn} \\ \text{Lilly} \end{matrix} \overset{\text{W1 W2}}{\begin{bmatrix} 1.50 & 1.35 \\ 0.95 & 1.00 \\ 1.30 & 1.35 \end{bmatrix}} = \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \end{matrix} \overset{\text{W1 W2}}{\begin{bmatrix} 18.40 & 18.15 \\ 21.60 & 20.85 \\ 21.05 & 20.50 \end{bmatrix}}$$

Figure 7.8 shows the product A^TB and supports our computation. Now try Exercise 47.

```
[A]T[B]
   [[18.4    18.15]
    [21.6    20.85]
    [21.05   20.5 ]]
```

FIGURE 7.8 The product A^TB for the matrices A and B of Example 5.

Identity and Inverse Matrices

The $n \times n$ matrix I_n with 1's on the main diagonal (upper left to lower right) and 0's elsewhere is the **identity matrix of order $n \times n$**

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If $A = [a_{ij}]$ is any $n \times n$ matrix, we can prove (see Exercise 56) that

$$AI_n = I_nA = A,$$

that is, I_n is the **multiplicative identity** for the set of $n \times n$ matrices.

If a is a nonzero real number, then $a^{-1} = 1/a$ is the multiplicative inverse of a, that is, $aa^{-1} = a(1/a) = 1$. The definition of the *multiplicative inverse* of a square matrix is similar.

Definition Inverse of a Square Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. If there is a matrix B such that

$$AB = BA = I_n,$$

then B is the **inverse** of A. We write $B = A^{-1}$ (read "A inverse").

We will see that not every square matrix (Example 7) has an inverse. If a square matrix A has an inverse, then A is **nonsingular**. If A has no inverse, then A is **singular**.

```
[A] [B]
            [[1 0]
             [0 1]]
[B] [A]     [[1 0]
             [0 1]]
```

FIGURE 7.9 Showing A and B are inverse matrices. (Example 6)

EXAMPLE 6 Verifying an inverse matrix

Prove that

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

are inverse matrices.

SOLUTION Figure 7.9 shows that $AB = BA = I_2$. Thus, $B = A^{-1}$ and $A = B^{-1}$.

Now try Exercise 33.

EXAMPLE 7 Showing a matrix has no inverse

Prove that the matrix $B = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$ is singular, that is, A has no inverse.

SOLUTION Suppose A has an inverse $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then, $AB = I_2$.

$$AB = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6x + 3z & 6y + 3w \\ 2x + z & 2y + w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using equality of matrices we obtain:

$$6x + 3z = 1 \qquad 6y + 3w = 0$$
$$2x + z = 0 \qquad 2y + w = 1$$

Multiplying both sides of the equation $2x + z = 0$ by 3 yields $6x + 3z = 0$. There are no values for x and z for which the value of $6x + 3z$ is both 0 and 1. Thus, A does not have an inverse.

Now try Exercise 37.

Determinant of a Square Matrix

There is a simple test that determines if a 2×2 matrix has an inverse.

Inverse of a 2 × 2 Matrix

If $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

The number $ad - bc$ is the **determinant** of the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and is denoted

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

To define the determinant of a higher order square matrix we need to introduce the *minors* and *cofactors* associated with the entries of a square matrix. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **minor** (short for "minor determinant") M_{ij} corresponding to the element a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting the row and column containing a_{ij}. The **cofactor** corresponding to a_{ij} is $A_{ij} = (-1)^{i+j}M_{ij}$.

Definition Determinant of a Square Matrix

Let $A = [a_{ij}]$ be a matrix of order $n \times n$ $(n > 2)$. The determinant of A, denoted by $\det A$ or $|A|$, is the sum of the entries in any row or any column multiplied by their respective cofactors. For example, expanding by the ith row gives

$$\det A = |A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

If $A = [a_{ij}]$ is a 3×3 matrix, then, using the definition of determinant applied to the second row, we obtain

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

$$= a_{21}(-1)^3\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(-1)^4\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{23}(-1)^5\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= -a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{22}(a_{11}a_{33} - a_{13}a_{31})$$
$$- a_{23}(a_{11}a_{32} - a_{12}a_{31})$$

The determinant of a 3×3 matrix involves three determinants of 2×2 matrices, the determinant of a 4×4 matrix involves four determinants of 3×3 matrices, and so forth. This is a tedious definition to apply. Most of the time we use a grapher to evaluate determinants in this textbook.

EXPLORATION 2 Investigating the Definition of Determinant

1. Complete the expansion of the determinant of the 3×3 matrix $A = [a_{ij}]$ started on the previous page. Explain why each term in the expansion contains an element from each row and each column.
2. Use the first row of the 3×3 matrix to expand the determinant and compare to the expression in 1.
3. Prove that the determinant of a square matrix with a zero row or a zero column is zero.

We can now state the condition under which square matrices have inverses.

Theorem Inverses of $n \times n$ Matrices

An $n \times n$ matrix A has an inverse if and only if $\det A \neq 0$.

There are complicated formulas for finding the inverses of nonsingular matrices of order 3×3 or higher. We will use a grapher instead of these formulas to find inverses of square matrices.

EXAMPLE 8 Finding inverse matrices

Determine whether the matrix has an inverse. If so, find its inverse matrix.

(a) $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ **(b)** $B = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ -1 & 0 & 1 \end{bmatrix}$

SOLUTION

(a) Since $\det A = ad - bc = 3 \cdot 2 - 1 \cdot 4 = 2 \neq 0$, we conclude that A has an inverse. Using the formula for the inverse of a 2×2 matrix, we obtain

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -0.5 \\ -2 & 1.5 \end{bmatrix}$$

You can check that $A^{-1}A = A^{-1}A = I_2$.

(b) Figure 7.10 shows that $\det B = -10 \neq 0$ and

$$B^{-1} = \begin{bmatrix} 0.1 & 0.2 & -0.5 \\ 0.5 & 0 & 0.5 \\ 0.1 & 0.2 & 0.5 \end{bmatrix}.$$

You can use your grapher to check that $B^{-1}B = BB^{-1} = I_3$.

Now try Exercise 41.

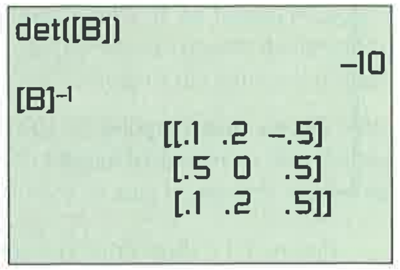


FIGURE 7.10 The matrix B is nonsingular and so has an inverse. (Example 8b)

We list some of the important properties of matrices, some of which you will be asked to prove in the exercises.

Properties of Matrices

Let A, B, and C be matrices whose orders are such that the following sums, differences, and products are defined.

1. Commutative property
Addition:
$A + B = B + A$
Multiplication:
(Does not hold in general)

2. Associative property
Addition:
$(A + B) + C = A + (B + C)$
Multiplication:
$(AB)C = A(BC)$

3. Identity property
Addition: $A + O = A$
Multiplication: order of $A = n \times n$
$A \cdot I_n = I_n \cdot A = A$

4. Inverse property
Addition: $A + (-A) = O$
Multiplication: order of $A = n \times n$
$AA^{-1} = A^{-1}A = I_n, \quad |A| \neq 0$

5. Distributive property

Multiplication over addition
$A(B + C) = AB + AC$
$(A + B)C = AC + BC$

Multiplication over subtraction
$A(B - C) = AB - AC$
$(A - B)C = AC - BC$

Applications

Points in the Cartesian coordinate plane can be represented by 1×2 matrices. For example, the point $(2, -3)$ can be represented by the 1×2 matrix $[2 \ -3]$. We can calculate the images of points acted upon by some of the transformations studied in Section 1.5 using matrix multiplication as illustrated in Example 9.

EXAMPLE 9 Reflecting with respect to the x-axis as matrix multiplication

Prove that the image of a point under a reflection across the x-axis can be obtained by multiplying by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

SOLUTION The image of the point (x, y) under a reflection across the x-axis is $(x, -y)$. The product

$$[x \quad y]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [x \quad -y]$$

shows that the point (x, y) (in matrix form $[x \quad y]$) is moved to the point $(x, -y)$ (in matrix form $[x \quad -y]$). Now try Exercise 57.

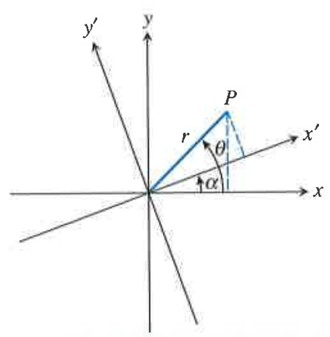


FIGURE 7.11 Rotating the xy-coordinate system through the angle α to obtain the $x'y'$-coordinate system. (Example 10)

Figure 7.11 shows the xy-coordinate system rotated through the angle α to obtain the $x'y'$-coordinate system. In Example 10, we see that the coordinates of a point in the $x'y'$-coordinate system can be obtained by multiplying the coordinates of the point in the xy-coordinate system by an appropriate 2×2 matrix. In Exercise 71, you will see that the reverse is also true.

EXAMPLE 10 Rotating a coordinate system

Prove that the (x', y') coordinates of P in Figure 7.11 are related to the (x, y) coordinates of P by the equations

$$x' = x \cos \alpha + y \sin \alpha$$

$$y' = -x \sin \alpha + y \cos \alpha.$$

Then, prove that the coordinates (x', y') can be obtained from the (x, y) coordinates by matrix multiplication. We use this result in Section 8.4 when we study conic sections.

SOLUTION Using the right triangle formed by P and the $x'y'$-coordinate system, we obtain

$$x' = r \cos(\theta - \alpha) \quad \text{and} \quad y' = r \sin(\theta - \alpha).$$

Expanding the above expressions for x' and y' using trigonometric identities for $\cos(\theta - \alpha)$ and $\sin(\theta - \alpha)$ yields

$$x' = r \cos\theta \cos\alpha + r \sin\theta \sin\alpha, \quad \text{and}$$

$$y' = r \sin\theta \cos\alpha - r \cos\theta \sin\alpha.$$

It follows from the right triangle formed by P and the xy-coordinate system that $x = r\cos\theta$ and $y = r\sin\theta$. Substituting these values for x and y into the above pair of equations yields

$$x' = x\cos\alpha + y\sin\alpha \quad \text{and} \quad y' = y\cos\alpha - x\sin\alpha$$

$$= -x\sin\alpha + y\cos\alpha,$$

which is what we were asked to prove. Finally, matrix multiplication shows that

$$[x' \quad y'] = [x \quad y]\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

Now try Exercise 71.

PROBLEM: If we have a triangle with vertices at (0, 0), (1, 1), and (2, 0), and we want to double the lengths of the sides of the triangle, where would the vertices of the enlarged triangle be?

SOLUTION: Given a triangle with vertices at (0, 0), (1, 1), and (2, 0), as in Figure 7.12, we can find the vertices of a new triangle whose sides are twice as long by multiplying by the scale matrix.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

For the point (0, 0), we have

$$[x' \quad y'] = [0 \quad 0]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [0 \quad 0].$$

For the point (1, 1), we have

$$[x' \quad y'] = [1 \quad 1]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [2 \quad 2].$$

And for the point (2, 0), we have

$$[x' \quad y'] = [2 \quad 0]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [4 \quad 0].$$

So the new triangle has vertices (0, 0), (2, 2), and (4, 0), as Figure 7.13 shows.

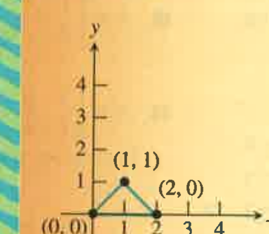


FIGURE 7.12

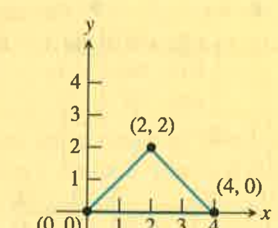


FIGURE 7.13

QUICK REVIEW 7.2

(For help, go to Sections 1.5, 5.3, and 6.4.)

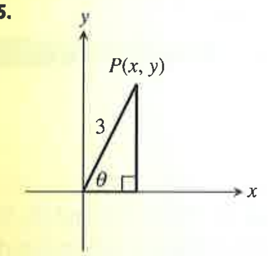
In Exercises 1–4, the points **(a)** $(3, -2)$ and **(b)** (x, y) are reflected across the given line. Find the coordinates of the reflected points.

1. The x-axis **2.** The y-axis

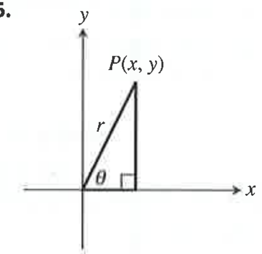
3. The line $y = x$ **4.** The line $y = -x$

In Exercises 5 and 6, express the coordinates of P in terms of θ.

5.



6.



In Exercises 7–10, expand the expression.

7. $\sin(\alpha + \beta)$ **8.** $\sin(\alpha - \beta)$

9. $\cos(\alpha + \beta)$ **10.** $\cos(\alpha - \beta)$

SECTION 7.2 EXERCISES

In Exercises 1–6, determine the order of the matrix. Indicate whether the matrix is square.

1. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 5 \end{bmatrix}$ **2.** $\begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}$ **3.** $\begin{bmatrix} 5 & 6 \\ -1 & 2 \\ 0 & 0 \end{bmatrix}$

4. $\begin{bmatrix} -1 & 0 & 6 \end{bmatrix}$ **5.** $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ **6.** $[\,0\,]$

In Exercises 7–10, identify the element specified for the following matrix.

$$\begin{bmatrix} -2 & 0 & 3 & 4 \\ 3 & 1 & 5 & -1 \\ 1 & 4 & -1 & 3 \end{bmatrix}$$

7. a_{13} **8.** a_{24} **9.** a_{32} **10.** a_{33}

In Exercises 11–16, find **(a)** $A + B$, **(b)** $A - B$, **(c)** $3A$, and **(d)** $2A - 3B$.

11. $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix}$

12. $A = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & -3 & -1 \end{bmatrix}$

13. $A = \begin{bmatrix} -3 & 1 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ -2 & 1 \\ -3 & -1 \end{bmatrix}$

14. $A = \begin{bmatrix} 5 & -2 & 3 & 1 \\ -1 & 0 & 2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 3 & 1 & 0 \\ 4 & 0 & -1 & -2 \end{bmatrix}$

15. $A = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$

16. $A = \begin{bmatrix} -1 & -2 & 0 & 3 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 2 & -2 & 0 \end{bmatrix}$

In Exercises 17–22, use the definition of matrix multiplication to find **(a)** AB and **(b)** BA. Support your answer with the matrix feature of your grapher.

17. $A = \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ -2 & -4 \end{bmatrix}$

18. $A = \begin{bmatrix} 1 & -4 \\ 2 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 1 \\ -2 & -3 \end{bmatrix}$

19. $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 4 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 0 & -2 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & 0 & -2 & 3 \\ 2 & 1 & 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ 0 & 2 \\ -1 & 3 \\ 4 & 2 \end{bmatrix}$

21. $A = \begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 2 \\ 4 & -3 & -1 \end{bmatrix}$

22. $A = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -2 & 4 \\ 3 & 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 2 & 3 \\ -1 & 3 & -1 \end{bmatrix}$

In Exercises 23–28, find **(a)** AB and **(b)** BA, or state that the product is not defined.

23. $A = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -5 \\ 4 \\ 2 \end{bmatrix}$

24. $A = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 4 \end{bmatrix}$

25. $A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 5 \end{bmatrix}$

26. $A = \begin{bmatrix} -1 & 3 \\ 0 & 1 \\ 1 & 0 \\ -3 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -6 \\ 2 & 3 \end{bmatrix}$

27. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

28. $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 & 3 & -4 \\ 2 & 1 & 0 & -1 \\ -3 & 2 & 1 & 3 \\ 4 & 0 & 2 & -1 \end{bmatrix}$

In Exercises 29–32, solve for a and b.

29. $\begin{bmatrix} a & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 4 & b \end{bmatrix}$

30. $\begin{bmatrix} 1 & -1 & 0 \\ a & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b & 0 \\ 3 & -2 & 1 \end{bmatrix}$

31. $\begin{bmatrix} 2 & a-1 \\ 2 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ b+2 & 3 \\ -1 & 2 \end{bmatrix}$

32. $\begin{bmatrix} a+3 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 0 & b-1 \end{bmatrix}$

In Exercises 33 and 34, verify that the matrices are inverses of each other.

33. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0.8 & -0.2 \\ -0.6 & 0.4 \end{bmatrix}$

34. $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & -2 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -2 \\ 0.25 & 0.5 & -0.25 \\ 0.25 & 0.5 & -1.25 \end{bmatrix}$

In Exercises 35–40, find the inverse of the matrix if it has one, or state that the inverse does not exist.

35. $\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$ **36.** $\begin{bmatrix} 6 & 3 \\ 10 & 5 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ **38.** $\begin{bmatrix} 2 & 3 & -1 \\ -1 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

39. $A = [a_{ij}]$, $a_{ij} = (-1)^{i+j}$, $1 \le i \le 4$, $1 \le j \le 4$

40. $B = [b_{ij}]$, $b_{ij} = |i - j|$, $1 \le i \le 3$, $1 \le j \le 3$

In Exercises 41 and 42, use the definition to evaluate the determinant of the matrix.

41. $\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & 3 & -1 \end{bmatrix}$ **42.** $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 0 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

In Exercises 43 and 44, solve for X.

43. $3X + A = B$, where $A = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

44. $2X + A = B$, where $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ 1 & -1 \end{bmatrix}$.

45. Symmetric Matrix The matrix below gives the road mileage between Atlanta (A), Baltimore (B), Cleveland (C), and Denver (D). (*Source: AAA Road Atlas*)

(a) Writing to Learn Explain why the entry in the ith row and jth column is the same as the entry in the jth row and ith column. A matrix with this property is **symmetric**.

(b) Writing to Learn Why are the entries along the diagonal all 0's?

	A	B	C	D
A	0	689	774	1406
B	689	0	371	1685
C	774	371	0	1340
D	1406	1685	1340	0

46. Production Jordan Manufacturing has two factories, each of which manufactures three products. The number of units of product i produced at factory j in one week is represented by a_{ij} in the matrix

$$A = \begin{bmatrix} 120 & 70 \\ 150 & 110 \\ 80 & 160 \end{bmatrix}.$$

If production levels are increased by 10%, write the new production levels as a matrix B. How is B related to A?

47. Egg Production Happy Valley Farms produces three types of eggs: 1 (large), 2 (X-large), 3 (jumbo). The number of dozens of type i eggs sold to grocery store j is represented by a_{ij} in the matrix.

$$A = \begin{bmatrix} 100 & 60 \\ 120 & 70 \\ 200 & 120 \end{bmatrix}.$$

The per dozen price Happy Valley Farms charges for egg type i is represented by b_{i1} in the matrix

$$B = \begin{bmatrix} \$0.80 \\ \$0.85 \\ \$1.00 \end{bmatrix}.$$

(a) Find the product B^TA.

(b) Writing to Learn What does the matrix B^TA represent?

48. Inventory A company sells four models of one name brand "all-in-one fax, printer, copier, and scanner machine" at three retail stores. The inventory at store i of model j is represented by s_{ij} in the matrix

$$S = \begin{bmatrix} 16 & 10 & 8 & 12 \\ 12 & 0 & 10 & 4 \\ 4 & 12 & 0 & 8 \end{bmatrix}.$$

The wholesale and retail prices of model i are represented by p_{i1} and p_{i2}, respectively, in the matrix

$$P = \begin{bmatrix} \$180 & \$269.99 \\ \$275 & \$399.99 \\ \$355 & \$499.99 \\ \$590 & \$799.99 \end{bmatrix}.$$

(a) Determine the product SP.

(b) Writing to Learn What does the matrix SP represent?

49. Profit A discount furniture store sells four types of 5-piece bedroom sets. The price charged for a bedroom set of type j is represented by a_{1j} in the matrix

$$A = [\$398 \quad \$598 \quad \$798 \quad \$998].$$

The number of sets of type j sold in one period is represented by b_{1j} in the matrix

$$B = [35 \quad 25 \quad 20 \quad 10].$$

The cost to the furniture store for a bedroom set of type j is given by c_{1j} in the matrix

$$C = [\$199 \quad \$268 \quad \$500 \quad \$670].$$

(a) Write a matrix product that gives the total revenue made from the sale of the bedroom sets in the one period.

(b) Write an expression using matrices that gives the profit produced by the sale of the bedroom sets in the one period.

50. Construction A building contractor has agreed to build six ranch-style houses, seven Cape Cod-style houses, and 14 colonial-style houses. The number of units of raw materials that go into each type of house are shown in the matrix

	Steel	Wood	Glass	Paint	Labor
Ranch	5	22	14	7	17
$R =$ Cape Cod	7	20	10	9	21
Colonial	6	27	8	5	13

Assume that steel costs \$1600 a unit, wood \$900 a unit, glass \$500 a unit, paint \$100 a unit, and labor \$1000 a unit.

(a) Write a 1×3 matrix B that represents the number of each type of house to be built.

(b) Write a matrix product that gives the number of units of each raw material needed to build the houses.

(c) Write a 5×1 matrix C that represents the per unit cost of each type of raw material.

(d) Write a matrix product that gives the cost of each house.

(e) Writing to Learn Compute the product BRC. What does this matrix represent?

51. Rotating Coordinate Systems The xy-coordinate system is rotated through the angle 30° to obtain the $x'y'$-coordinate system.

(a) If the coordinates of a point in the xy-coordinate system are (1, 1), what are the coordinates of the rotated point in the $x'y'$-coordinate system?

(b) If the coordinates of a point in the $x'y'$-coordinate system are (1, 1), what are the coordinates of the point in the xy-coordinate system that was rotated to it?

52. Group Activity Let A, B, and C be matrices whose orders are such that the following expressions are defined. Prove that the following properties are true.

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $A(B + C) = AB + AC$

(d) $(A - B)C = AC - BC$

53. Group Activity Let A and B be $m \times n$ matrices and c and d scalars. Prove that the following properties are true.

(a) $c(A + B) = cA + cB$ **(b)** $(c + d)A = cA + dA$

(c) $c(dA) = (cd)A$ **(d)** $1 \cdot A = A$

54. Writing to Learn Explain why the definition given for the determinant of a square matrix agrees with the definition given for the determinant of a 2×2 matrix. (Assume that the determinant of a 1×1 matrix is the entry.)

55. Inverse of a 2 × 2 Matrix Prove that the inverse of the matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided $ad - bc \neq 0$.

56. Identity Matrix Let $A = [a_{ij}]$ be an $n \times n$ matrix. Prove that $AI_n = I_nA = A$.

In Exercises 57–61, prove that the image of a point under the given transformation of the plane can be obtained by matrix multiplication.

57. A reflection across the y-axis

58. A reflection across the line $y = x$

59. A reflection across the line $y = -x$

60. A vertical stretch or shrink by a factor of a

61. A horizontal stretch or shrink by a factor of c

Standardized Test Questions

62. True or False Every square matrix has an inverse. Justify your answer.

63. True or False The determinant $|A|$ of the square matrix A is greater than or equal to 0. Justify your answer.

In Exercises 64–67, solve the problem without using a calculator.

64. Multiple Choice Which of the following is equal to the determinant of $A = \begin{bmatrix} 2 & 4 \\ -3 & -1 \end{bmatrix}$?

(a) 4 **(b)** −4 **(c)** 10 **(d)** −10 **(e)** −14

65. Multiple Choice Let A be a matrix of order 3×2 and B a matrix of order 2×4. Which of the following gives the order of the product AB?

(a) 2×2 **(b)** 3×4 **(c)** 4×3 **(d)** 6×8

(e) The product is not defined.

66. Multiple Choice Which of the following is the inverse of the matrix $\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$?

(a) $\begin{bmatrix} -4 & 7 \\ 1 & -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ **(c)** $\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ **(e)** $\begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix}$

67. Multiple Choice Which of the following is the value of a_{13} in the matrix $[a_{ij}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$?

(a) −7 **(b)** 7 **(c)** −3 **(d)** 3 **(e)** 10

Explorations

68. Continuation of Exploration 2 Let $A = [a_{ij}]$ be an $n \times n$ matrix.

(a) Prove that the determinant of A changes sign if two rows or two columns are interchanged. Start with a 3×3 matrix and compare the expansion by expanding by the same row (or column) before and after the interchange. [*Hint:* Compare without expanding the minors.] How can you generalize from the 3×3 case?

(b) Prove that the determinant of a square matrix with two identical rows or two identical columns is zero.

(c) Prove that if a scalar multiple of a row (or column) is added to another row (or column) the value of the determinant of a square matrix is unchanged. [*Hint:* Expand by the row (or column) being added to.]

69. Continuation of Exercise 68 Let $A = [a_{ij}]$ be an $n \times n$ matrix.

(a) Prove that if every element of a row or column of a matrix is multiplied by the real number c, then the determinant of the matrix is multiplied by c.

(b) Prove that if all the entries above the main diagonal (or all below it) of a matrix are zero, the determinant is the product of the elements on the main diagonal.

70. Writing Equations for Lines Using Determinants Consider the equation

$$\begin{vmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{vmatrix} = 0.$$

(a) Verify that the equation is linear in x and y.

(b) Verify that the two points (x_1, y_1) and (x_2, y_2) lie on the line in (a).

(c) Use a determinant to state that the point (x_3, y_3) lies on the line in (a).

(d) Use a determinant to state that the point (x_3, y_3) does not lie on the line in (a).

71. Continuation of Example 10 The xy-coordinate system is rotated through the angle α to obtain the $x'y'$-coordinate system (see Figure 7.11).

(a) Show that the inverse of the matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

of Example 10 is

$$A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}.$$

(b) Prove that the (x, y) coordinates of P in Figure 7.11 are related to the (x', y') coordinates of P by the equations

$$x = x' \cos\alpha - y' \sin\alpha$$
$$y = x' \sin\alpha + y' \cos\alpha.$$

(c) Prove that the coordinates (x, y) can be obtained from the (x', y') coordinates by matrix multiplication. How is this matrix related to A?

Extending the Ideas

72. Characteristic Polynomial Let $A = [a_{ij}]$ be a 2×2 matrix and define $f(x) = \det(xI_2 - A)$.

(a) Expand the determinant to show that $f(x)$ is a polynomial of degree 2. (The **characteristic polynomial** of A.)

(b) How is the constant term of $f(x)$ related to det A?

(c) How is the coefficient of x related to A?

(d) Prove that $f(A) = 0$.

73. Characteristic Polynomial Let $A = [a_{ij}]$ be a 3×3 matrix and define $f(x) = \det(xI_3 - A)$.

(a) Expand the determinant to show that $f(x)$ is a polynomial of degree 3. (The characteristic polynomial of A.)

(b) How is the constant term of $f(x)$ related to det A?

(c) How is the coefficient of x^2 related to A?

(d) Prove that $f(A) = 0$.

7.3 MULTIVARIATE LINEAR SYSTEMS AND ROW OPERATIONS

What you'll learn about

- Triangular Form for Linear Systems
- Gaussian Elimination
- Elementary Row Operations and Row Echelon Form
- Reduced Row Echelon Form
- Solving Systems with Inverse Matrices
- Applications

. . . and why

Many applications in business and science are modeled by systems of linear equations in three or more variables.

Triangular Form for Linear Systems

The method of elimination used in Section 7.1 can be extended to systems of linear (first-degree) equations in more than two variables. The goal of the elimination method is to rewrite the system as an *equivalent system* of equations whose solution is obvious. Two systems of equations are **equivalent** if they have the same solution.

A *triangular form* of a system is an equivalent form from which the solution is easy to read. Here is an example of a system in triangular form.

$$x - 2y + z = 7$$
$$y - 2z = -7$$
$$z = 3$$

This convenient triangular form allows us to solve the system using substitution as illustrated in Example 1.

EXAMPLE 1 Solving by substitution

Solve the system

$$x - 2y + z = 7$$
$$y - 2z = -7$$
$$z = 3.$$

SOLUTION The third equation determines z, namely $z = 3$. Substitute the value of z into the second equation to determine y.

$$y - 2z = -7 \quad \text{Second equation}$$
$$y - 2(3) = -7 \quad \text{Substitute } z = 3.$$
$$y = -1$$

Finally, substitute the values for y and z into the first equation to determine x.

$$x - 2y + z = 7 \quad \text{First equation}$$
$$x - 2(-1) + 3 = 7 \quad \text{Substitute } y = -1, z = 3.$$
$$x = 2$$

The solution of the system is $x = 2$, $y = -1$, $z = 3$, or the ordered triple $(2, -1, 3)$. Now try Exercise 1.

BACK SUBSTITUTION

The method of solution used in Example 1 is sometimes referred to as *back substitution*.

Gaussian Elimination

Transforming a system to triangular form is **Gaussian elimination**, named after the famous German mathematician Carl Friedrich Gauss (1777–1855).

Here are the operations needed to transform a system of linear equations into triangular form.

Equivalent Systems of Linear Equations

The following operations produce an equivalent system of linear equations.

1. Interchange any two equations of the system.
2. Multiply (or divide) one of the equations by any nonzero real number.
3. Add a multiple of one equation to any other equation in the system.

Watch how we use property 3 to bring the system in Example 2 to triangular form.

EXAMPLE 2 Using Gaussian elimination

Solve the system

$$x - 2y + z = 7$$
$$3x - 5y + z = 14$$
$$2x - 2y - z = 3.$$

SOLUTION Each step in the following process leads to a system of equations equivalent to the original system.

Multiply the first equation by -3 and add the result to the second equation, replacing the second equation. (Leave the first and third equations unchanged.)

$$x - 2y + z = 7$$
$$y - 2z = -7 \qquad \begin{aligned} -3x + 6y - 3z &= -21 \\ 3x - 5y + z &= 14 \end{aligned}$$
$$2x - 2y - z = 3$$

Multiply the first equation by -2 and add the result to the third equation, replacing the third equation.

$$x - 2y + z = 7$$
$$y - 2z = -7$$
$$2y - 3z = -11 \qquad \begin{aligned} -2x + 4y - 2z &= -14 \\ 2x - 2y - z &= 3 \end{aligned}$$

Multiply the second equation by -2 and add the result to the third equation, replacing the third equation.

$$x - 2y + z = 7$$
$$y - 2z = -7$$
$$z = 3 \qquad \begin{aligned} -2y + 4z &= 14 \\ 2y - 3z &= -11 \end{aligned}$$

This is the same system of Example 1 and is a triangular form of the original system. We know from Example 1 that the solution is $(2, -1, 3)$. Now try Exercise 3.

For a system of equations which has exactly one solution, the final system in Example 2 is in **triangular form**. In this case, the leading term of each equation has coefficient 1, the third equation has one variable (z), the second equation has at most two variables including one not in the third equation (y), and the first one has the remaining variable, x in this case.

EXAMPLE 3 Finding no solution

Solve the system

$$\begin{aligned} x - 3y + z &= 4 \\ -x + 2y - 5z &= 3 \\ 5x - 13y + 13z &= 8. \end{aligned}$$

SOLUTION Use Gaussian elimination.

$$\begin{aligned} x - 3y + z &= 4 \\ -y - 4z &= 7 \\ 5x - 13y + 13z &= 8 \end{aligned}$$

Add 1st equation to 2nd equation.

$$\begin{aligned} x - 3y + z &= 4 \\ -y - 4z &= 7 \\ 2y + 8z &= -12 \end{aligned}$$

Multiply 1st equation by -5 and add to 3rd equation.

$$\begin{aligned} x - 3y + z &= 4 \\ -y - 4z &= 7 \\ 0 &= 2 \end{aligned}$$

Multiply 2nd equation by 2 and add to 3rd equation.

Since $0 = 2$ is never true, we conclude that this system has no solution.

Now try Exercise 5.

Elementary Row Operations and Row Echelon Form

When we solve a system of linear equations using Gaussian elimination, all the action is really on the coefficients of the variables. Matrices can be used to record the coefficients as we go through the steps of the Gaussian elimination process. We illustrate with the system of Example 2.

$$\begin{aligned} x - 2y + z &= 7 \\ 3x - 5y + z &= 14 \\ 2x - 2y - z &= 3 \end{aligned}$$

The **augmented matrix** of this system of equations is

$$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 3 & -5 & 1 & 14 \\ 2 & -2 & -1 & 3 \end{bmatrix}.$$

The entries in the last column are the numbers on the right-hand side of the equations. For the record, the **coefficient matrix** of this system is

$$\begin{bmatrix} 1 & -2 & 1 \\ 3 & -5 & 1 \\ 2 & -2 & -1 \end{bmatrix}.$$

Here the entries are the coefficients of the variables. We use this matrix to solve certain linear systems later in this section.

We repeat the Gaussian elimination process used in Example 2 and record the corresponding action on the augmented matrix.

System of Equations	Augmented Matrix	
$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ 2x - 2y - z &= 3 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 2 & -2 & -1 & 3 \end{bmatrix}$	Multiply Eq. 1 (Row 1) by -3, add result to Eq. 2 (Row 2) replacing Eq. 2 (Row 2).
$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ 2y - 3z &= -11 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 2 & -3 & -11 \end{bmatrix}$	Multiply Eq. 1 (Row 1) by -2, add result to Eq. 3 (Row 3) replacing Eq. 3 (Row 3).
$\begin{aligned} x - 2y + z &= 7 \\ y - 2z &= -7 \\ z &= 3 \end{aligned}$	$\begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$	Multiply Eq. 2 (Row 2) by -2, add result to Eq. 3 (Row 3) replacing Eq. 3 (Row 3).

The augmented matrix above, corresponding to the triangular form of the original system of equations, is a *row echelon form* of the augmented matrix of the original system of equations. In general, the last few rows of a row echelon form of a matrix *can* consist of all 0's. We will see examples like this in a moment.

Definition Row Echelon Form of a Matrix

A matrix is in **row echelon form** if the following conditions are satisfied.

1. Rows consisting entirely of 0's (if there are any) occur at the bottom of the matrix.
2. The first entry in any row with nonzero entries is 1.
3. The column subscript of the leading 1 entries increases as the row subscript increases.

Another way to phrase parts 2 and 3 of the above definition is to say that the leading 1's move to the right as we move down the rows.

Our goal is to take a system of equations, write the corresponding augmented matrix, and transform it to row echelon form without carrying along the equations. From there we can read off the solutions to the system fairly easily.

The operations that we use to transform a linear system to equivalent triangular form correspond to elementary row operations of the corresponding augmented matrix of the linear system.

Elementary Row Operations on a Matrix

A combination of the following operations will transform a matrix to row echelon form.

1. Interchange any two rows.
2. Multiply all elements of a row by a nonzero real number.
3. Add a multiple of one row to any other row.

Example 4 illustrates how we can transform the augmented matrix to row echelon form to solve a system of linear equations.

EXAMPLE 4 Finding a row echelon form

Solve the system

$$x - y + 2z = -3$$
$$2x + y - z = 0$$
$$-x + 2y - 3z = 7.$$

SOLUTION We apply elementary row operations to find a row echelon form of the augmented matrix. The elementary row operations used are recorded above the arrows using the notation in the margin.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 1 & -1 & 0 \\ -1 & 2 & -3 & 7 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ -1 & 2 & -3 & 7 \end{bmatrix} \xrightarrow{(1)R_1+R_3}$$

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 3 & -5 & 6 \\ 0 & 1 & -1 & 4 \end{bmatrix} \xrightarrow{R_{23}} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 3 & -5 & 6 \end{bmatrix} \xrightarrow{(-3)R_2+R_3}$$

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & -2 & -6 \end{bmatrix} \xrightarrow{(-1/2)R_3} \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is in row echelon form. Then we convert each row into equation form and complete the solution by substitution.

NOTATION

1. R_{ij} indicates interchanging the ith and jth row of a matrix.
2. kR_i indicates multiplying the ith row by the nonzero real number k.
3. $kR_i + R_j$ indicates adding k times the ith row to the jth row.

ROW ECHELON FORM

A word of caution! You can use your grapher to find a row echelon form of a matrix. However, row echelon form is *not* unique. Your grapher may produce a row echelon form different from the one you obtained by paper-and-pencil. Fortunately, all row echelon forms produce the same solution to the system of equations. (Correspondingly, a triangular form of a linear system is also not unique.)

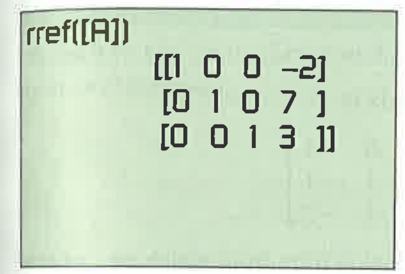


FIGURE 7.14 *A* is the augmented matrix of the system of linear equations in Example 4. "rref" stands for the grapher-produced reduced row echelon form of *A*.

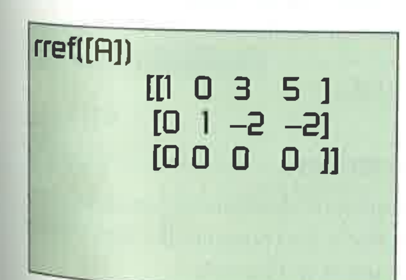


FIGURE 7.15 The reduced row echelon form for the augmented matrix of Example 5.

$$\begin{array}{lll} & y - z = 4 & x - y + 2z = -3 \\ z = 3 & y - 3 = 4 & x - 7 + 2(3) = -3 \\ & y = 7 & x = -2 \end{array}$$

The solution of the original system of equations is $(-2, 7, 3)$.

Now try Exercise 33.

Reduced Row Echelon Form

If we continue to apply elementary row operations to a row echelon form of a matrix, we can obtain a matrix in which every column that has a leading 1 has 0's elsewhere. This is the **reduced row echelon form** of the matrix. It is usually easier to read the solution from the reduced row echelon form.

We apply elementary row operations to the row echelon form found in Example 4 until we find the reduced row echelon form.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(1)R_2+R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(-1)R_3+R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(1)R_3+R_2} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

From this reduced row echelon form, we can immediately read the solution to the system of Example 4: $x = -2$, $y = 7$, $z = 3$. Figure 7. 14 shows that the above final matrix is the reduced row echelon form of the augmented matrix of Example 4.

EXAMPLE 5 Finding infinitely many solutions

Solve the system

$$x + y + z = 3$$
$$2x + y + 4z = 8$$
$$x + 2y - z = 1.$$

SOLUTION Figure 7.15 shows the reduced row echelon form for the augmented matrix of the system. So, the following system of equations is equivalent to the original system.

$$x + 3z = 5$$
$$y - 2z = -2$$
$$0 = 0$$

Solving the first two equations for x and y in terms of z yields:

$$x = -3z + 5$$
$$y = 2z - 2.$$

This system has infinitely many solutions because for every value of z we can use these two equations to find corresponding values for x and y.

Interpret

The solution is the set of all ordered triples of the form $(-3z + 5, 2z - 2, z)$ where z is any real number. Now try Exercise 39.

We can also solve linear systems with more than three variables, or more than three equations, or both, by finding a row (or reduced row) echelon form. The solution set may become more complicated as illustrated in Example 6.

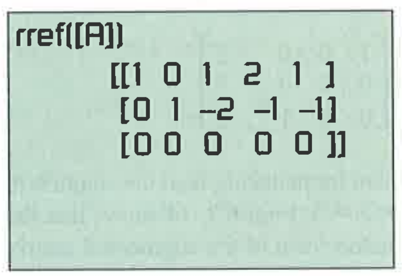


FIGURE 7.16 The reduced row echelon form for the augmented matrix of Example 6.

EXAMPLE 6 Finding infinitely many solutions

Solve the system

$$\begin{aligned} x + 2y - 3z \qquad &= -1 \\ 2x + 3y - 4z + w &= -1 \\ 3x + 5y - 7z + w &= -2. \end{aligned}$$

SOLUTION The 3×5 augmented matrix is

$$\begin{bmatrix} 1 & 2 & -3 & 0 & -1 \\ 2 & 3 & -4 & 1 & -1 \\ 3 & 5 & -7 & 1 & -2 \end{bmatrix}.$$

Figure 7.16 shows the reduced row echelon form from which we can read that

$$\begin{aligned} x &= -z - 2w + 1 \\ y &= 2z + w - 1. \end{aligned}$$

This system has infinitely many solutions because for every pair of values for z and w we can use these two equations to find corresponding values for x and y.

Interpret

The solution is the set of all ordered 4-tuples of the form $(-z - 2w + 1, 2z + w - 1, z, w)$ where z and w are any real numbers. Now try Exercise 43.

Solving Systems with Inverse Matrices

If a linear system consists of the same number of equations as variables, then the coefficient matrix is square. If this matrix is also nonsingular, then we can solve the system using the technique illustrated in Example 7.

EXAMPLE 7 Solving a system using inverse matrices

Solve the system

$$\begin{aligned} 3x - 2y &= 0 \\ -x + y &= 5. \end{aligned}$$

LINEAR EQUATIONS

If a and b are real numbers with $a \neq 0$, the linear equation $ax = b$ has a unique solution $x = a^{-1}b$. A similar statement holds for the linear matrix equation $AX = B$ when A is a nonsingular square matrix. (See the Invertible Square Linear Systems Theorem.)

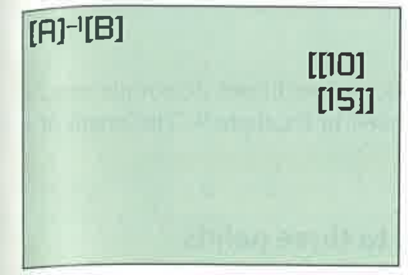


FIGURE 7.17 The solution of the matrix equation of Example 7.

SOLUTION First we write the system as a matrix equation. Let

$$A = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

Then

$$A \cdot X = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x - 2y \\ -x + y \end{bmatrix}$$

so that

$$AX = B,$$

where A is the coefficient matrix of the system. You can easily check that $\det A = 1$, so A^{-1} exists. From Figure 7.17, we obtain

$$X = A^{-1}B = \begin{bmatrix} 10 \\ 15 \end{bmatrix}.$$

The solution of the system is $x = 10$, $y = 15$, or $(10, 15)$. Now try Exercise 49.

Examples 7 and 8 are two instances of the following theorem.

Theorem Invertible Square Linear Systems

Let A be the coefficient matrix of a system of n linear equations in n variables given by $AX = B$, where X is the $n \times 1$ matrix of variables and B is the $n \times 1$ matrix of numbers on the right-hand side of the equations. If A^{-1} exists, then the system of equations has the unique solution

$$X = A^{-1}B.$$

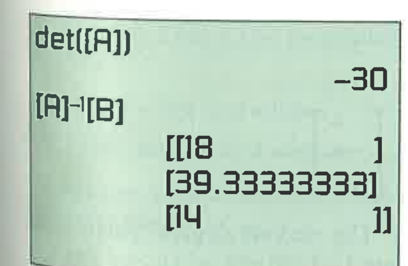


FIGURE 7.18 The solution of the system in Example 8.

EXAMPLE 8 Solving a system using inverse matrices

Solve the system

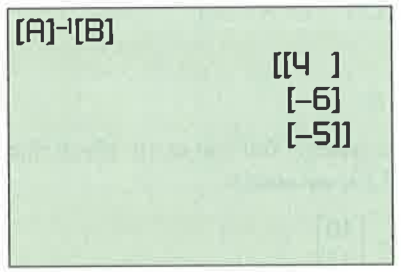
$$\begin{aligned} 3x - 3y + 6z &= 20 \\ x - 3y + 10z &= 40 \\ -x + 3y - 5z &= 30. \end{aligned}$$

SOLUTION Let

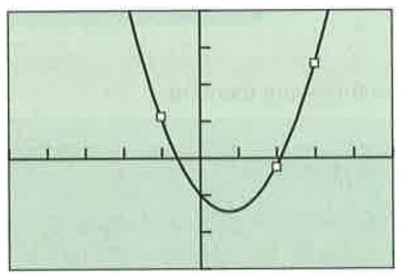
$$A = \begin{bmatrix} 3 & -3 & 6 \\ 1 & -3 & 10 \\ -1 & 3 & -5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 20 \\ 40 \\ 30 \end{bmatrix}.$$

The system of equations can be written as

$$A \cdot X = B.$$



(a)



[−5, 5] by [−15, 20]
(b)

FIGURE 7.19 (a) The solution of the matrix equation of Example 9. (b) A graph of $f(x) = 4x^2 - 6x - 5$ superimposed on a scatter plot of the three points $(-1, 5)$, $(2, -1)$, and $(3, 13)$.

Figure 7.18 shows that $\det A = -30 \neq 0$, so A^{-1} exists and

$$X = A^{-1}B = \begin{bmatrix} 18 \\ 39.\overline{3} \\ 14 \end{bmatrix}.$$

Interpret

The solution of the system of equations is $x = 18$, $y = 39\frac{1}{3}$, and $z = 14$, or $(18, 39\frac{1}{3}, 14)$. Now try Exercise 51.

Applications

Any three noncollinear points with distinct x-coordinates determine exactly one second-degree polynomial as illustrated in Example 9. The graph of a second-degree polynomial is a parabola.

EXAMPLE 9 Fitting a parabola to three points

Determine a, b, and c so that the points $(-1, 5)$, $(2, -1)$, and $(3, 13)$ are on the graph of $f(x) = ax^2 + bx + c$.

SOLUTION

Model

We must have $f(-1) = 5$, $f(2) = -1$, and $f(3) = 13$:

$$f(-1) = a - b + c = 5$$
$$f(2) = 4a + 2b + c = -1$$
$$f(3) = 9a + 3b + c = 13.$$

The above system of three linear equations in the three variables a, b, and c can be written in matrix form $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 5 \\ -1 \\ 13 \end{bmatrix}.$$

Solve Numerically

Figure 7.19a shows that

$$X = A^{-1}B = \begin{bmatrix} 4 \\ -6 \\ -5 \end{bmatrix}.$$

Thus, $a = 4$, $b = -6$, and $c = -5$. The second-degree polynomial $f(x) = 4x^2 - 6x - 5$ contains the points $(-1, 5)$, $(2, -1)$, and $(3, 13)$ (Figure 7.19b). Now try Exercise 67.

EXPLORATION 1 Mixing Solutions

Aileen's Drugstore needs to prepare a 60-L mixture that is 40% acid using three concentrations of acid. The first concentration is 15% acid, the second is 35% acid, and the third is 55% acid. Because of the amounts of acid solution on hand, they need to use twice as much of the 35% solution as the 55% solution. How much of each solution should they use?

Let x = the number of liters of 15% solution used, y = the number of liters of 35% solution used, and z = the number of liters of 55% solution used.

1. Explain how the equation $x + y + z = 60$ is related to the problem.
2. Explain how the equation $0.15x + 0.35y + 0.55z = 24$ is related to the problem.
3. Explain how the equation $y = 2z$ is related to the problem.
4. Write the system of three equations obtained from parts 1–3 in the form $AX = B$, where A is the coefficient matrix of the system. What are A, B, and X?
5. Solve the matrix equation in part 4.
6. Interpret the solution in part 5 in terms of the problem situation.

QUICK REVIEW 7.3

(For help, go to Sections 1.2 and 7.2.)

In Exercises 1 and 2, find the amount of pure acid in the solution.

1. 40 L of a 32% acid solution
2. 60 mL of a 14% acid solution

In Exercises 3 and 4, find the amount of water in the solution.

3. 50 L of a 24% acid solution
4. 80 mL of a 70% acid solution

In Exercises 5 and 6, determine which points are on the graph of the function.

5. $f(x) = 2x^2 - 3x + 1$
   (a) $(-1, 6)$ **(b)** $(2, 1)$
6. $f(x) = x^3 - 4x - 1$
   (a) $(0, -1)$ **(b)** $(-2, -17)$

In Exercises 7 and 8, solve for x or y in terms of the other variables.

7. $y + z - w = 1$
8. $x - 2z + w = 3$

In Exercises 9 and 10, find the inverse of the matrix.

9. $\begin{bmatrix} 1 & 3 \\ -2 & -2 \end{bmatrix}$
10. $\begin{bmatrix} 0 & 0 & 2 \\ -2 & 1 & 3 \\ 0 & 2 & -2 \end{bmatrix}$

SECTION 7.3 EXERCISES

In Exercises 1 and 2, use substitution to solve the system of equations.

1. $$\begin{aligned} x - 3y + z &= 0 \\ 2y + 3z &= 1 \\ z &= -2 \end{aligned}$$

2. $$\begin{aligned} 3x - y + 2z &= -2 \\ y + 3z &= 3 \\ 2z &= 4 \end{aligned}$$

In Exercises 3–8, use Gaussian elimination to solve the system of equations.

3. $$\begin{aligned} x - y + z &= 0 \\ 2x - 3z &= -1 \\ -x - y + 2z &= -1 \end{aligned}$$

4. $$\begin{aligned} 2x - y &= 0 \\ x + 3y - z &= -3 \\ 3y + z &= 8 \end{aligned}$$

5. $$\begin{aligned} x + y + z &= -3 \\ 4x - y &= -5 \\ -3x + 2y + z &= 4 \end{aligned}$$

6. $$\begin{aligned} x + y - 3z &= -1 \\ 2x - 3y + z &= 4 \\ 3x - 7y + 5z &= 4 \end{aligned}$$

7. $$\begin{aligned} x + y - z &= 4 \\ y + w &= -4 \\ x - y &= 1 \\ x + z + w &= 1 \end{aligned}$$

8. $$\begin{aligned} \frac{1}{2}x - y + z - w &= 1 \\ -x + y + z + 2w &= -3 \\ x - z &= 2 \\ y + w &= 0 \end{aligned}$$

In Exercises 9–12, perform the indicated elementary row operation on the matrix

$$\begin{bmatrix} 2 & -6 & 4 \\ 1 & 2 & -3 \\ -3 & 1 & -2 \end{bmatrix}.$$

9. $(3/2)R_1 + R_3$

10. $(1/2)R_1$

11. $(-2)R_2 + R_1$

12. $(1)R_1 + R_2$

In Exercises 13–16, what elementary row operations applied to

$$\begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

will yield the given matrix?

13. $$\begin{bmatrix} 1 & -2 & 3 & 0 \\ -2 & 1 & -1 & 2 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

14. $$\begin{bmatrix} 0 & -3 & 5 & 2 \\ 1 & -2 & 3 & 0 \\ 3 & 1 & -1 & 2 \end{bmatrix}$$

15. $$\begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 0 & 7 & -10 & 2 \end{bmatrix}$$

16. $$\begin{bmatrix} -2 & 1 & -1 & 2 \\ 1 & -2 & 3 & 0 \\ 0.75 & 0.25 & -0.25 & 0.5 \end{bmatrix}$$

In Exercises 17–20, find a row echelon form for the matrix.

17. $$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$$

18. $$\begin{bmatrix} 1 & 2 & -3 \\ -3 & -6 & 10 \\ -2 & -4 & 7 \end{bmatrix}$$

19. $$\begin{bmatrix} 1 & 2 & 3 & -4 \\ -2 & 6 & -6 & 2 \\ 3 & 12 & 6 & 12 \end{bmatrix}$$

20. $$\begin{bmatrix} 3 & 6 & 9 & -6 \\ 2 & 5 & 5 & -3 \end{bmatrix}$$

In Exercises 21–24, find the reduced row echelon form for the matrix.

21. $$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 3 & 2 & 4 & 7 \\ 2 & 1 & 3 & 4 \end{bmatrix}$$

22. $$\begin{bmatrix} 1 & -2 & 2 & 1 & 1 \\ 3 & -5 & 6 & 3 & -1 \\ -2 & 4 & -3 & -2 & 5 \\ 3 & -5 & 6 & 4 & -3 \end{bmatrix}$$

23. $$\begin{bmatrix} 1 & 2 & 3 & 1 \\ -3 & -5 & -7 & -4 \end{bmatrix}$$

24. $$\begin{bmatrix} 3 & -6 & 3 & -3 \\ 2 & -4 & 2 & -2 \\ -3 & 6 & -3 & 3 \end{bmatrix}$$

In Exercises 25–28, write the augmented matrix corresponding to the system of equations.

25. $$\begin{aligned} 2x - 3y + z &= 1 \\ -x + y - 4z &= -3 \\ 3x - z &= 2 \end{aligned}$$

26. $$\begin{aligned} 3x - 4y + z - w &= 1 \\ x + z - 2w &= 4 \end{aligned}$$

27. $$\begin{aligned} 2x - 5y + z - w &= -3 \\ x - 2z + w &= 4 \\ 2y - 3z - w &= 5 \end{aligned}$$

28. $$\begin{aligned} 3x - 2y &= 5 \\ -x + 5y &= 7 \end{aligned}$$

In Exercises 29–32, write the system of equations corresponding to the augmented matrix.

29. $$\begin{bmatrix} 3 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$$

30. $$\begin{bmatrix} 1 & 0 & -1 & 2 & -3 \\ 2 & 1 & 0 & -1 & 4 \\ -1 & 1 & 2 & 0 & 0 \end{bmatrix}$$

31. $$\begin{bmatrix} 2 & 0 & 1 & 3 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

32. $$\begin{bmatrix} 2 & 1 & -2 & 4 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

In Exercises 33–34, solve the system of equations by finding a row echelon form for the augmented matrix.

33. $$\begin{aligned} x - 2y + z &= 8 \\ 2x + y - 3z &= -9 \\ -3x + y + 3z &= 5 \end{aligned}$$

34. $$\begin{aligned} 3x + 7y - 11z &= 44 \\ x + 2y - 3z &= 12 \\ 4x + 9y - 13z &= 53 \end{aligned}$$

In Exercises 35–44, solve the system of equations by finding the reduced row echelon form for the augmented matrix.

35. $$\begin{aligned} x + 2y - z &= 3 \\ 3x + 7y - 3z &= 12 \\ -2x - 4y + 3z &= -5 \end{aligned}$$

36. $$\begin{aligned} x - 2y + z &= -2 \\ 2x - 3y + 2z &= 2 \\ 4x - 8y + 5z &= -5 \end{aligned}$$

37. $$\begin{aligned} x + y + 3z &= 2 \\ 3x + 4y + 10z &= 5 \\ x + 2y + 4z &= 3 \end{aligned}$$

38. $$\begin{aligned} x - z &= 2 \\ -2x + y + 3z &= -5 \\ 2x + y - z &= 3 \end{aligned}$$

39. $$\begin{aligned} x + z &= 2 \\ 2x + y + z &= 5 \end{aligned}$$

40. $$\begin{aligned} x + 2y - 3z &= 1 \\ -3x - 5y + 8z &= -29 \end{aligned}$$

41. $$\begin{aligned} x + 2y &= 4 \\ 3x + 4y &= 5 \\ 2x + 3y &= 4 \end{aligned}$$

42. $$\begin{aligned} x + y &= 3 \\ 2x + 3y &= 8 \\ 2x + 2y &= 6 \end{aligned}$$

43. $$\begin{aligned} x + y - 3z &= 1 \\ x - z - w &= 2 \\ 2x + y - 4z - w &= 3 \end{aligned}$$

44. $$\begin{aligned} x - y - z + 2w &= -3 \\ 2x - y - 2z + 3w &= -3 \\ x - 2y - z + 3w &= -6 \end{aligned}$$

In Exercises 45 and 46, write the system of equations as a matrix equation $AX = B$, with A as the coefficient matrix of the system.

45. $$\begin{aligned} 2x + 5y &= -3 \\ x - 2y &= 1 \end{aligned}$$

46. $$\begin{aligned} 5x - 7y + z &= 2 \\ 2x - 3y - z &= 3 \\ x + y + z &= -3 \end{aligned}$$

In Exercises 47 and 48, write the matrix equation as a system of equations.

47. $$\begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

48. $$\begin{bmatrix} 1 & 0 & -3 \\ 2 & -1 & 3 \\ -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

In Exercises 49–54, solve the system of equations by using an inverse matrix.

49. $$\begin{aligned} 2x - 3y &= -13 \\ 4x + y &= -5 \end{aligned}$$

50. $$\begin{aligned} x + 2y &= -2 \\ 3x - 4y &= 9 \end{aligned}$$

51. $$\begin{aligned} 2x - y + z &= -6 \\ x + 2y - 3z &= 9 \\ 3x - 2y + z &= -3 \end{aligned}$$

52. $$\begin{aligned} x + 4y - 2z &= 0 \\ 2x + y + z &= 6 \\ -3x + 3y - 5z &= -13 \end{aligned}$$

53. $$\begin{aligned} 2x - y + z + w &= -3 \\ x + 2y - 3z + w &= 12 \\ 3x - y - z + 2w &= 3 \\ -2x + 3y + z - 3w &= -3 \end{aligned}$$

54. $$\begin{aligned} 2x + y + 2z &= 8 \\ 3x + 2y - z - w &= 10 \\ -2x + y - 3w &= -1 \\ 4x - 3y + 2z - 5w &= 39 \end{aligned}$$

In Exercises 55–66, use a method of your choice to solve the system of equations.

55. $$\begin{aligned} 2x - y &= 10 \\ x - z &= -1 \\ y + z &= -9 \end{aligned}$$

56. $$\begin{aligned} 1.25x + z &= -2 \\ y - 5.5z &= -2.75 \\ 3x - 1.5y &= -6 \end{aligned}$$

57. $$\begin{aligned} x + 2y + 2z + w &= 5 \\ 2x + y + 2z &= 5 \\ 3x + 3y + 3z + 2w &= 12 \\ x + z + w &= 1 \end{aligned}$$

58. $$\begin{aligned} x - y + w &= -4 \\ -2x + y + z &= 8 \\ 2x - 2y - z &= -10 \\ -2x + z + w &= 5 \end{aligned}$$

59. $$\begin{aligned} x - y + z &= 6 \\ x + y + 2z &= -2 \end{aligned}$$

60. $$\begin{aligned} x - 2y + z &= 3 \\ 2x + y - z &= -4 \end{aligned}$$

61. $$\begin{aligned} 2x + y + z + 4w &= -1 \\ x + 2y + z + w &= 1 \\ x + y + z + 2w &= 0 \end{aligned}$$

62. $$\begin{aligned} 2x + 3y + 3z + 7w &= 0 \\ x + 2y + 2z + 5w &= 0 \\ x + y + 2z + 3w &= -1 \end{aligned}$$

63. $$\begin{aligned} 2x + y + z + 2w &= -3.5 \\ x + y + z + w &= -1.5 \end{aligned}$$

64. $$\begin{aligned} 2x + y + 4w &= 6 \\ x + y + z + w &= 5 \end{aligned}$$

65. $$\begin{aligned} x + y - z + 2w &= 0 \\ y - z + 2w &= -1 \\ x + y + 3w &= 3 \\ 2x + 2y - z + 5w &= 4 \end{aligned}$$

66. $$\begin{aligned} x + y + w &= 2 \\ x + 4y + z - 2w &= 3 \\ x + 3y + z - 3w &= 2 \\ x + y + w &= 2 \end{aligned}$$

In Exercises 67–70, determine f so that its graph contains the given points.

67. Curve Fitting $f(x) = ax^2 + bx + c$
$(-1, 3), (1, -3), (2, 0)$

68. Curve Fitting $f(x) = ax^3 + bx^2 + cx + d.$
$(-2, -37), (-1, -11), (0, -5), (2, 19)$

69. Family of Curves $f(x) = ax^2 + bx + c$
$(-1, -4), (1, -2)$

70. Family of Curves $f(x) = ax^3 + bx^2 + cx + d$
$(-1, -6), (0, -1), (1, 2)$

71. Population Table 7.5 gives the population (in thousands) for Corpus Christi, TX, and Garland, TX, for several years. Use $x = 0$ for 1970, $x = 1$ for 1971, and so forth.

TABLE 7.5 POPULATION

Year	Corpus Christi (thousands)	Garland (thousands)
1970	205	81
1980	232	139
1990	258	181
2000	277	216

Source: U. S. Census Bureau, Statistical Abstract of the United States, 2001.

(a) Find the linear regression equation for the Corpus Christi data and superimpose its graph on a scatter plot of the data.

(b) Find the linear regression equation for the Garland data and superimpose its graph on a scatter plot of the data.

(c) Estimate when the population of the two cities will be the same.

72. Population Table 7.6 gives the population (in thousands) for Anaheim, CA, and Anchorage, AK, for several years. Use $x = 0$ for 1970, $x = 1$ for 1971, and so forth.

TABLE 7.6 POPULATION

Year	Anaheim (thousands)	Anchorage (thousands)
1970	166	48
1980	219	174
1990	267	226
2000	328	260

Source: U.S. Census Bureau, Statistical Abstract of the United States, 2001.

(a) Find the linear regression equation for the Anaheim data and superimpose its graph on a scatter plot of the data.

(b) Find the linear regression equation for the Anchorage data and superimpose its graph on a scatter plot of the data.

(c) Estimate when the population of the two cities will be the same.

73. Train Tickets At the Pittsburgh zoo, children ride a train for 25 cents, adults pay \$1.00, and senior citizens 75 cents. On a given day, 1400 passengers paid a total of \$740 for the rides. There were 250 more children riders than all other riders. Find the number of children, adult, and senior riders.

74. Manufacturing Stewart's Metals has three silver alloys on hand. One is 22% silver, another is 30% silver, and the third is 42% silver. How many grams of each alloy is required to produce 80 grams of a new alloy that is 34% silver if the amount of 30% alloy used is twice the amount of 22% alloy used.

75. Investment Monica receives an \$80,000 inheritance. She invests part of it in CDs (certificates of deposit) earning 6.7% APY (annual percentage yield), part in bonds earning 9.3% APY, and the remainder in a growth fund earning 15.6% APY. She invests three times as much in the growth fund as in the other two combined. How much does she have in each investment if she receives \$10,843 interest the first year?

76. Investments Oscar invests \$20,000 in three investments earning 6% APY, 8% APY, and 10% APY. He invests \$9000 more in the 10% investment than in the 6% investment. How much does he have invested at each rate if he receives \$1780 interest the first year?

77. Investments Morgan has \$50,000 to invest and wants to receive \$5000 interest the first year. He puts part in CDs earning 5.75% APY, part in bonds earning 8.7% APY, and the rest in a growth fund earning 14.6% APY. How much should he invest at each rate if he puts the least amount possible in the growth fund?

78. Mixing Acid Solutions Simpson's Drugstore needs to prepare a 40-L mixture that is 32% acid from three solutions: a 10% acid solution, a 25% acid solution, and a 50% acid solution. How much of each solution should be used if Simpson's wants to use as little of the 50% solution as possible?

79. Loose Change Matthew has 74 coins consisting of nickels, dimes, and quarters in his coin box. The total value of the coins is \$8.85. If the number of nickels and quarters is four more than the number of dimes, find how many of each coin Matthew has in his coin box.

80. Vacation Money Heather has saved \$177 to take with her on the family vacation. She has 51 bills consisting of \$1, \$5, and \$10 bills. If the number of \$5 bills is three times the number of \$10 bills, find how many of each bill she has.

In Exercises 81–82, use inverse matrices to find the equilibrium point for the demand and supply curves.

81. $p = 100 - 5x$ Demand curve
$p = 20 + 10x$ Supply curve

82. $p = 150 - 12x$ Demand curve
$p = 30 + 24x$ Supply curve

83. Writing to Learn Explain why adding one row to another row in a matrix is an elementary row operation.

84. Writing to Learn Explain why subtracting one row from another row in a matrix is an elementary row operation.

Standardized Test Questions

85. True or False Every nonzero square matrix has an inverse. Justify your answer.

86. True or False The reduced row echelon form of the augmented matrix of a system of three linear equations in three variables must be of the form

$$\begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix},$$

where a, b, c, are real numbers. Justify your answer.

In Exercises 87–90, you may use a graphing calculator to solve the problem.

87. Multiple Choice Which of the following is the determinant of the matrix $\begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$?

(a) 0 **(b)** 4
(c) -4 **(d)** 8
(e) -8

88. Multiple Choice Which of the following is the augmented matrix of the system of equations

$$\begin{aligned} x + 2y + z &= -1 \\ 2x - y + 3z &= -4 \\ 3x + y - z &= -2 \end{aligned}?$$

(a) $\begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & -1 & 3 & -4 \\ 3 & 1 & -1 & -2 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 3 & 1 & -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & 1 & -1 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & -3 \\ 3 & 1 & 1 \end{bmatrix}$

89. Multiple Choice The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$ was obtained from $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ by an elementary row operation. Which of the following describes the elementary row operation?

(a) $(-2)R_1$ **(b)** $(-2)R_1 + R_2$
(c) $(-2)R_2 + R_1$ **(d)** $(2)R_1 + R_2$
(e) $(2)R_2 + R_1$

90. Multiple Choice Which of the following is the reduced row echelon form for the augmented matrix of

$$\begin{aligned} x + 2y - z &= 8 \\ -x + 3y + 2z &= 3 \\ 2x - y + 3z &= -19 \end{aligned}?$$

(a) $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -4 \end{bmatrix}$

Explorations

91. Group Activity Investigating the Solution of a System of 3 Linear Equations in 3 Variables Assume that the graph of a linear equation in three variables is a plane in 3-dimensional space. (You will study these in Chapter 8.)

(a) Explain geometrically how such a system can have a unique solution.

(b) Explain geometrically how such a system can have no solution. Describe several possibilities.

(c) Explain geometrically how such a system can have infinitely many solutions. Describe several possibilities. Construct physical models if you find that helpful.

Extending the Ideas

92. Writing to Learn Explain why a row echelon form of a matrix is not unique. That is, show that a matrix can have two unequal row echelon forms. Give an example.

The roots of the characteristic polynomial $C(x) = \det(xI_n - A)$ of the $n \times n$ matrix A are the **eigenvalues** of A (see Section 7.2, Exercises 72 and 73). Use this information in Exercises 93 and 94.

93. Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$.

(a) Find the characteristic polynomial $C(x)$ of A.

(b) Find the graph of $y = C(x)$.

(c) Find the eigenvalues of A.

(d) Compare det A with the y-intercept of the graph of $y = C(x)$.

(e) Compare the sum of the main diagonal elements of A $(a_{11} + a_{22})$ with the sum of the eigenvalues.

94. Let $A = \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix}$.

(a) Find the characteristic polynomial $C(x)$ of A.

(b) Find the graph of $y = C(x)$.

(c) Find the eigenvalues of A.

(d) Compare det A with the y-intercept of the graph of $y = C(x)$.

(e) Compare the sum of the main diagonal elements of A $(a_{11} + a_{22})$ with the sum of the eigenvalues.