CHAPTER 8

Analytic Geometry in Two and Three Dimensions

The oval-shaped lawn behind the White House in Washington, D.C. is called *the Ellipse*. It has views of the Washington Monument, the Jefferson Memorial, the Department of Commerce, and the Old Post Office Building. *The Ellipse* is 616 ft long, 528 ft wide, and is in the shape of a conic section. Its shape can be modeled using the methods of this chapter. See page 651.

History of Conic Sections

Parabolas, ellipses, and hyperbolas had been studied for many years when Apollonius (c. 262–190 B.C.) wrote his eight-volume *Conic Sections*. Apollonius, born in northwestern Asia Minor, was the first to unify these three curves as cross sections of a cone and to view the hyperbola as having two branches. Interest in conic sections was renewed in the 17th century when Galileo proved that projectiles follow parabolic paths and Johannes Kepler (1571–1630) discovered that planets travel in elliptical orbits.

Chapter 8 Overview

Analytic geometry combines number with form. It is the marriage of algebra and geometry that grew from the works of Frenchmen René Descartes (1596–1650) and Pierre de Fermat (1601–1665). Their achievements allowed geometry problems to be solved algebraically and algebra problems to be solved geometrically—two major themes of this book. Analytic geometry opened the door for Newton and Leibniz to develop calculus.

In Sections 8.1–8.4, we will learn that parabolas, ellipses, and hyperbolas are all conic sections and can all be expressed as second-degree equations. We will investigate their uses, including the reflective properties of parabolas and ellipses and how hyperbolas are used in long-range navigation. In Section 8.5, we will see how parabolas, ellipses, and hyperbolas are unified in the polar-coordinate setting. In Section 8.6, we will move from the two-dimensional plane to revisit the concepts of point, line, midpoint, distance, and vector in three-dimensional space.

8.1 CONIC SECTIONS AND PARABOLAS

What you'll learn about

- Conic Sections
- Geometry of a Parabola
- Translations of Parabolas
- Reflective Property of a Parabola

. . . and why

Conic sections are the paths of nature: Any free-moving object in a gravitational field follows the path of a conic section.

Conic Sections

Imagine two nonperpendicular lines intersecting at a point V. If we fix one of the lines as an *axis* and rotate the other line (the *generator*) around the axis, then the generator sweeps out a **right circular cone** with **vertex** V, as illustrated in Figure 8.1. Notice that V divides the cone into two parts called **nappes**, with each nappe of the cone resembling a pointed ice-cream cone.

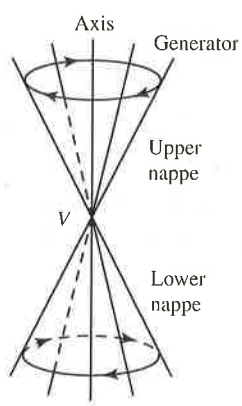


FIGURE 8.1 A right circular cone (of two nappes).

A **conic section** (or **conic**) is a cross section of a cone, in other words, the intersection of a plane with a right circular cone. The three basic conic sections are the *parabola*, the *ellipse*, and the *hyperbola* (Figure 8.2a).

Some atypical conics, known as **degenerate conic sections**, are shown in Figure 8.2b. Because it is atypical and lacks some of the features usually

associated with an ellipse, a circle is considered to be a degenerate ellipse. Other degenerate conic sections can be obtained from cross sections of a degenerate cone; such cones occur when the generator and axis of the cone are parallel or perpendicular (see Exercise 73).

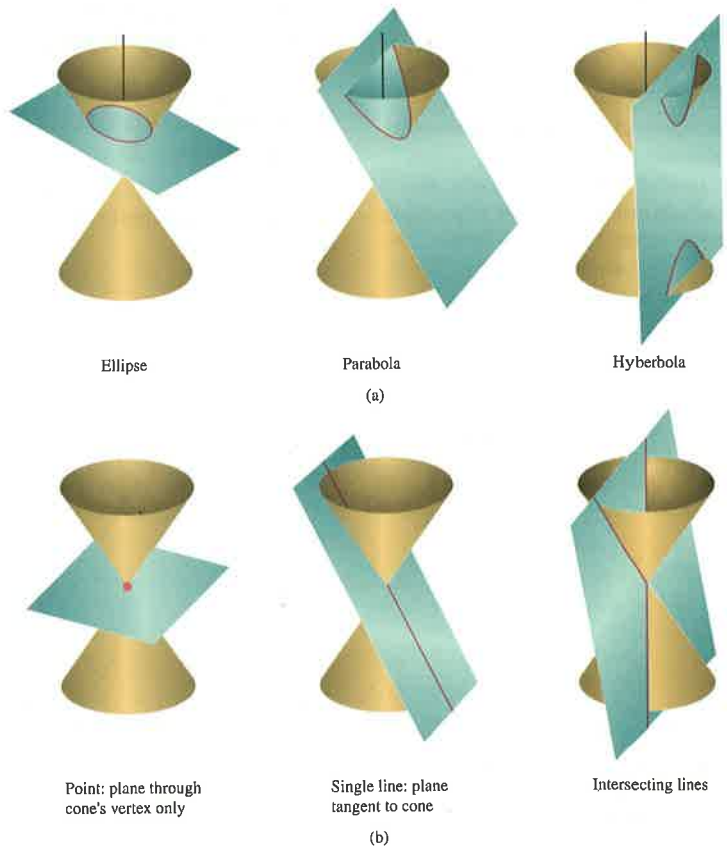


FIGURE 8.2 (a) The three standard types of conic sections and (b) three degenerate conic sections.

The conic sections can be defined algebraically as the graphs of **second-degree (quadratic) equations in two variables**, that is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero.

Geometry of a Parabola

In Section 2.1 we learned that the graph of a quadratic function is an upward or downward opening parabola. We have seen the role of the parabola in free-fall and projectile motion. We now investigate the geometric properties of parabolas.

A Degenerate Parabola

If the focus F lies on the directrix l, the parabola "degenerates" to the line through F perpendicular to l. Henceforth, we will assume F does not lie on l.

Locus of a Point

Before the word *set* was used in mathematics, the Latin word *locus*, meaning "place," was often used in geometric definitions. The locus of a point was the set of possible places a point could be and still fit the conditions of the definition. Sometimes, conics are still defined in terms of loci.

Definition Parabola

A **parabola** is the set of all points in a plane equidistant from a particular line (the **directrix**) and a particular point (the **focus**) in the plane. (See Figure 8.3.)

The line passing through the focus and perpendicular to the directrix is the focal **axis** of the parabola. The axis is the line of symmetry for the parabola. The point where the parabola intersects its axis is the **vertex** of the parabola. The vertex is located midway between the focus and the directrix and is the point of the parabola that is closest to both the focus and the directrix. See Figure 8.3.

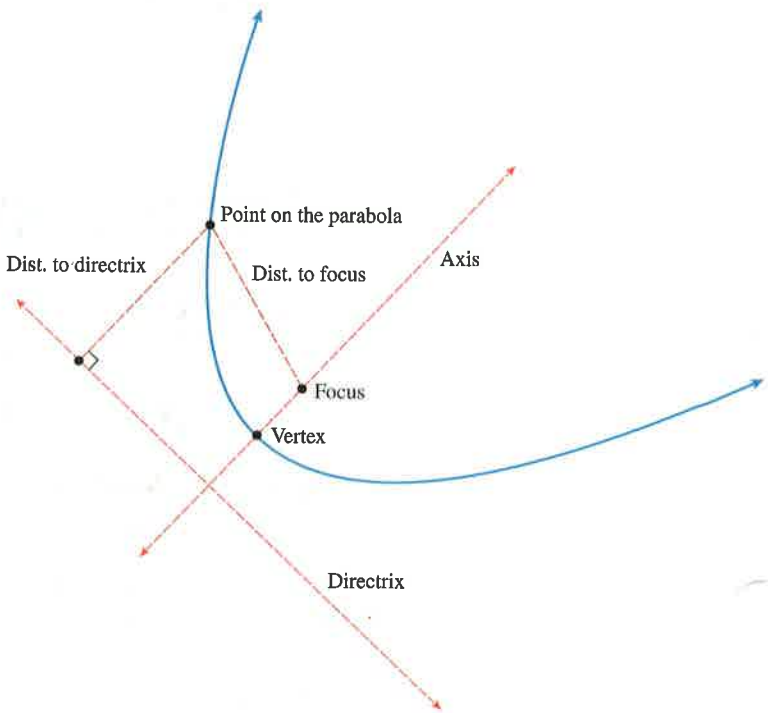


FIGURE 8.3 *Structure of a Parabola.* The distance from each point on the parabola to both the focus and the directrix is the same.

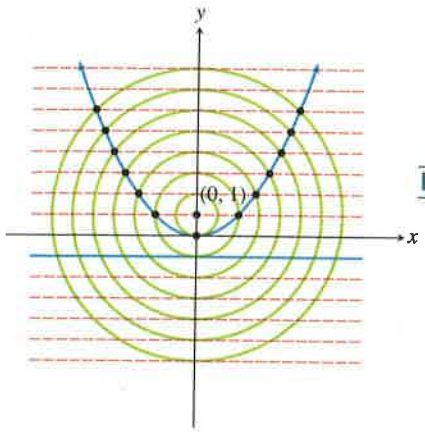


FIGURE 8.4 The geometry of a parabola.

EXPLORATION 1 Understanding the Definition of Parabola

1. Prove that the vertex of the parabola with focus $(0, 1)$ and directrix $y = -1$ is $(0, 0)$. (See Figure 8.4.)
2. Find an equation for the parabola shown in Figure 8.4.
3. Find the coordinates of the points of the parabola that are highlighted in Figure 8.4.

We can generalize the situation in Exploration 1 to show that an equation for the parabola with focus $(0, p)$ and directrix $y = -p$ is $x^2 = 4py$. (See Figure 8.5.)

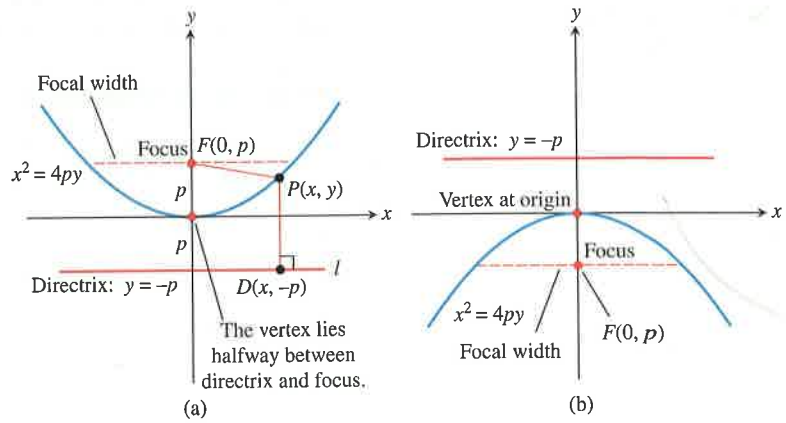


FIGURE 8.5 Graphs of $x^2 = 4py$ with (a) $p > 0$ and (b) $p < 0$.

We must show first that a point $P(x, y)$ that is equidistant from $F(0, p)$ and the line $y = -p$ satisfies the equation $x^2 = 4py$, and then that a point satisfying the equation $x^2 = 4py$ is equidistant from $F(0, p)$ and the line $y = -p$:

Let $P(x, y)$ be equidistant from $F(0, p)$ and the line $y = -p$. Then

$$\sqrt{(x-0)^2 + (y-p)^2} = \text{distance from } P(x, y) \text{ to } F(0, p), \text{ and}$$

$$\sqrt{(x-x)^2 + (y-(-p))^2} = \text{distance from } P(x, y) \text{ to } y = -p.$$

Equating these distances and squaring yields:

$$(x-0)^2 + (y-p)^2 = (x-x)^2 + (y-(-p))^2$$

$$x^2 + (y-p)^2 = 0 + (y+p)^2 \qquad \text{Simplify.}$$

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2 \qquad \text{Expand.}$$

$$x^2 = 4py \qquad \text{Combine like terms.}$$

By reversing the above steps, we see that a solution (x, y) of $x^2 = 4py$ is equidistant from $F(0, p)$ and the line $y = -p$.

The equation $x^2 = 4py$ is the **standard form** of the equation of an upward or downward opening parabola with vertex at the origin. If $p > 0$, the parabola opens upward; if $p < 0$, it opens downward. An alternative algebraic form for such a parabola is $y = ax^2$, where $a = 1/(4p)$. So the graph of $x^2 = 4py$ is also the graph of the quadratic function $f(x) = ax^2$.

When the equation of an upward or downward opening parabola is written as $x^2 = 4py$, the value p is interpreted as the **focal length** of the parabola—the *directed* distance from the vertex to the focus of the parabola. A line segment with endpoints on a parabola is a **chord** of the parabola. The value $|4p|$ is the **focal width** of the parabola—the length of the chord through the focus and perpendicular to the axis.

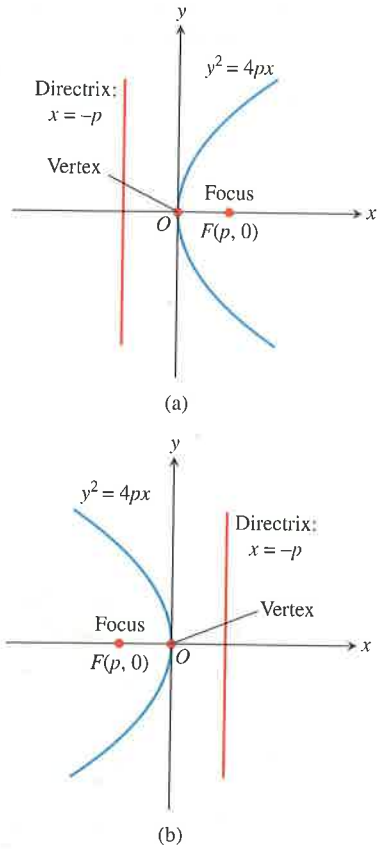


FIGURE 8.6 Graph of $y^2 = 4px$ with (a) $p > 0$ and (b) $p < 0$.

Parabolas that open to the right or to the left are *inverse relations* of upward or downward opening parabolas. So equations of parabolas with vertex (0, 0) that open to the right or to the left have the standard form $y^2 = 4px$. If $p > 0$, the parabola opens to the right, and if $p < 0$, to the left. (See Figure 8.6.)

Parabolas with Vertex (0, 0)

• **Standard equation**	$x^2 = 4py$	$y^2 = 4px$
• **Opens**	Upward or downward	To the right or to the left
• **Focus**	$(0, p)$	$(p, 0)$
• **Directrix**	$y = -p$	$x = -p$
• **Axis**	y-axis	x-axis
• **Focal length**	p	p
• **Focal width**	$\|4p\|$	$\|4p\|$

See Figures 8.5 and 8.6.

EXAMPLE 1 Finding the focus, directrix, and focal width

Find the focus, the directrix, and the focal width of the parabola $y = -(1/2)x^2$.

SOLUTION Multiplying both sides of the equation by -2 yields the standard form $x^2 = -2y$. The coefficient of y is $4p = -2$, and $p = -1/2$. So the focus is $(0, p) = (0, -1/2)$. Because $-p = -(-1/2) = 1/2$, the directrix is the line $y = 1/2$. The focal width is $|4p| = |-2| = 2$.

Now try Exercise 1.

EXAMPLE 2 Finding an equation of a parabola

Find an equation in standard form for the parabola whose directrix is the line $x = 2$ and whose focus is the point $(-2, 0)$.

SOLUTION Because the directrix is $x = 2$ and the focus is $(-2, 0)$, the focal length is $p = -2$ and the parabola opens to the left. The equation of the parabola in standard form is $y^2 = 4px$, or more specifically, $y^2 = -8x$.

Now try Exercise 15.

Translations of Parabolas

When a parabola with the equation $x^2 = 4py$ or $y^2 = 4px$ is translated horizontally by h units and vertically by k units, the vertex of the parabola moves from (0, 0) to (h, k). (See Figure 8.7.) Such a translation does not change the focal length, the focal width, or the direction the parabola opens.

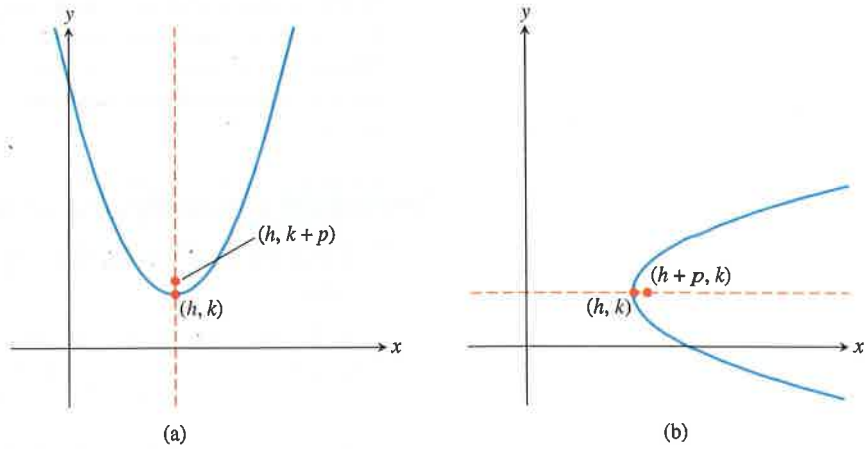


FIGURE 8.7 Parabolas with vertex (h, k) and focus on (a) $x = h$ and (b) $y = k$.

Parabolas with Vertex (h, k)

• **Standard equation**	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
• **Opens**	Upward or downward	To the right or to the left
• **Focus**	$(h, k + p)$	$(h + p, k)$
• **Directrix**	$y = k - p$	$x = h - p$
• **Axis**	$x = h$	$y = k$
• **Focal length**	p	p
• **Focal width**	$\lvert 4p \rvert$	$\lvert 4p \rvert$

See Figure 8.7.

EXAMPLE 3 Finding an equation of a parabola

Find the standard form of the equation for the parabola with vertex (3, 4) and focus (5, 4).

SOLUTION The axis of the parabola is the line passing through the vertex (3, 4) and the focus (5, 4). This is the line $y = 4$. So the equation has the form

$$(y - k)^2 = 4p(x - h).$$

Because the vertex $(h, k) = (3, 4)$, $h = 3$ and $k = 4$. The directed distance from the vertex (3, 4) to the focus (5, 4) is $p = 5 - 3 = 2$, so $4p = 8$. Thus the equation we seek is

$$(y - 4)^2 = 8(x - 3).$$

Now try Exercise 21.

When solving a problem like Example 3, it is a good idea to sketch the vertex, the focus, and other features of the parabola as we solve the problem. This makes it easy to see whether the axis of the parabola is horizontal or vertical and the relative positions of its features. Exploration 2 "walks us through" this process.

EXPLORATION 2 Building a Parabola

Carry out the following steps using a sheet of rectangular graph paper.

1. Let the focus F of a parabola be $(2, -2)$ and its directrix be $y = 4$. Draw the x- and y-axes on the graph paper. Then sketch and label the focus and directrix of the parabola.
2. Locate, sketch, and label the axis of the parabola. What is the equation of the axis?
3. Locate and plot the vertex V of the parabola. Label it by name and coordinates.
4. What are the focal length and focal width of the parabola?
5. Use the focal width to locate, plot, and label the endpoints of a chord of the parabola that parallels the directrix.
6. Sketch the parabola.
7. Which direction does it open?
8. What is its equation in standard form?

Sometimes it is best to sketch a parabola by hand, as in Exploration 2; this helps us see the structure and relationships of the parabola and its features. At other times, we may want or need an accurate, active graph. If we wish to graph a parabola using a function grapher, we need to solve the equation of the parabola for y, as illustrated in Example 4.

EXAMPLE 4 Graphing a parabola

Use a function grapher to graph the parabola $(y - 4)^2 = 8(x - 3)$ of Example 3.

SOLUTION

$$(y - 4)^2 = 8(x - 3)$$

$$y - 4 = \pm\sqrt{8(x - 3)} \qquad \text{Extract square roots.}$$

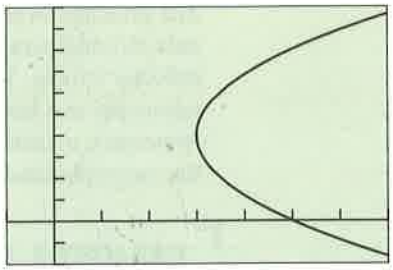
$$y = 4 \pm \sqrt{8(x - 3)} \qquad \text{Add .}$$

Let $Y_1 = 4 + \sqrt{8(x - 3)}$ and $Y_2 = 4 - \sqrt{8(x - 3)}$, and graph the two equations in a window centered at the vertex, as shown in Figure 8.8.

Now try Exercise 45.

Closing the Gap

In Figure 8.8, we centered the graphing window at the vertex (3, 4) of the parabola to ensure that this point would be plotted. This avoids the common grapher error of a gap between the two upper and lower parts of the conic section being plotted.



[–1, 7] by [–2, 10]

FIGURE 8.8 The graphs of $Y1 = 4 + \sqrt{8(x-3)}$ and $Y2 = 4 - \sqrt{8(x-3)}$ together form the graph of $(y-4)^2 = 8(x-3)$. (Example 4)

EXAMPLE 5 Using standard forms with a parabola

Prove that the graph of $y^2 - 6x + 2y + 13 = 0$ is a parabola, and find its vertex, focus, and directrix.

SOLUTION Because this equation is quadratic in the variable y, we complete the square with respect to y to obtain a standard form.

$$y^2 - 6x + 2y + 13 = 0$$

$$y^2 + 2y = 6x - 13 \qquad \text{Isolate the } y\text{-terms.}$$

$$y^2 + 2y + 1 = 6x - 13 + 1 \qquad \text{Complete the square.}$$

$$(y+1)^2 = 6x - 12$$

$$(y+1)^2 = 6(x-2)$$

This equation is in the standard form $(y-k)^2 = 4p(x-h)$, where $h = 2$, $k = -1$, and $p = 6/4 = 3/2 = 1.5$. It follows that

- the vertex (h, k) is $(2, -1)$;
- the focus $(h + p, k)$ is $(3.5, -1)$, or $(7/2, -1)$;
- the directrix $x = h - p$ is $x = 0.5$, or $x = 1/2$.

Now try Exercise 49.

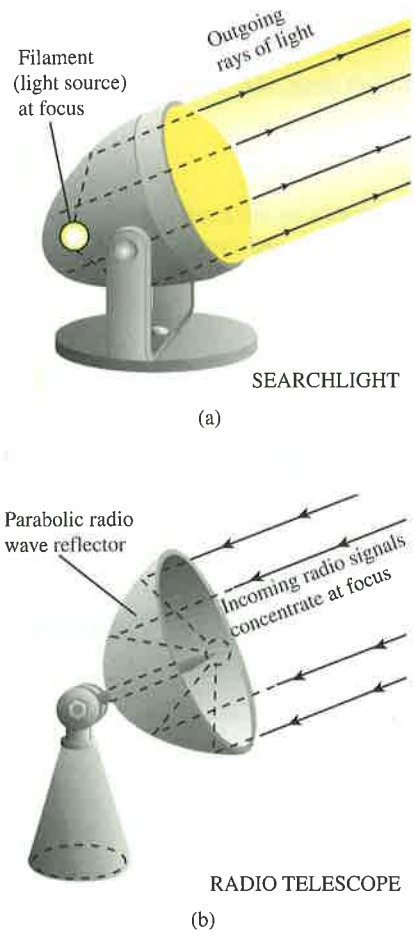


FIGURE 8.9 Examples of parabolic reflectors.

Reflective Property of a Parabola

The main applications of parabolas involve their use as reflectors of sound, light, radio waves, and other electromagnetic waves. If we rotate a parabola in three-dimensional space about its axis, the parabola sweeps out a **paraboloid of revolution**. If we place a signal source at the focus of a reflective paraboloid, the signal reflects off the surface in lines parallel to the axis of symmetry, as illustrated in Figure 8.9a. This property is used by flashlights, headlights, searchlights, microwave relays, and satellite up-links.

The principle works for signals traveling in the reverse direction as well; signals arriving parallel to a parabolic reflector's axis are directed toward the reflector's focus. This property is used to intensify signals picked up by radio telescopes and television satellite dishes, to focus arriving light in reflecting telescopes, to concentrate heat in solar ovens, and to magnify sound for sideline microphones at football games. See Figure 8.9b.

EXAMPLE 6 Studying a parabolic microphone

On the sidelines of each of its televised football games, the FBTV network uses a parabolic reflector with a microphone at the reflector's focus to capture the conversations among players on the field. If the parabolic reflector is 3 ft across and 1 ft deep, where should the microphone be placed?

SOLUTION

Model

We draw a cross section of the reflector as an upward opening parabola in the Cartesian plane, placing its vertex V at the origin (see Figure 8.10). We let the focus F have coordinates $(0, p)$ to yield the equation

$$x^2 = 4py.$$

Because the reflector is 3 ft across and 1 ft deep, the points $(\pm 1.5, 1)$ must lie on the parabola.

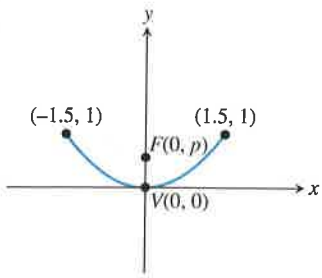


FIGURE 8.10 Cross section of parabolic reflector in Example 6.

Solve Algebraically

The microphone should be placed at the focus, so we need to find the value of p. We do this by substituting the values we found into the equation:

$$\begin{aligned} x &= 4py \\ (\pm 1.5)^2 &= 4p(1) \\ 2.25 &= 4p \\ p &= 2.25/4 = 0.5625 \end{aligned}$$

Interpret

Because $p = 0.5625$ ft, or 6.75 inches, the microphone should be placed inside the reflector along its axis and 6.75 inches from its vertex.

Now try Exercise 59.

QUICK REVIEW 8.1

(For help, go to Sections P.2, P.5, and 2.1.)

In Exercises 1 and 2, find the distance between the given points.

1. $(-1, 3)$ and $(2, 5)$ **2.** $(2, -3)$ and (a, b)

In Exercises 3 and 4, solve for y in terms of x.

3. $2y^2 = 8x$ **4.** $3y^2 = 15x$

In Exercises 5 and 6, complete the square to rewrite the equation in vertex form.

5. $y = -x^2 + 2x - 7$ **6.** $y = 2x^2 + 6x - 5$

In Exercises 7 and 8, find the vertex and axis of the graph of f. Describe how the graph of f can be obtained from the graph of $g(x) = x^2$, and graph f.

7. $f(x) = 3(x - 1)^2 + 5$ **8.** $f(x) = -2x^2 + 12x + 1$

In Exercises 9 and 10, write an equation for the quadratic function whose graph contains the given vertex and point.

9. Vertex $(-1, 3)$, point $(0, 1)$

10. Vertex $(2, -5)$, point $(5, 13)$

SECTION 8.1 EXERCISES

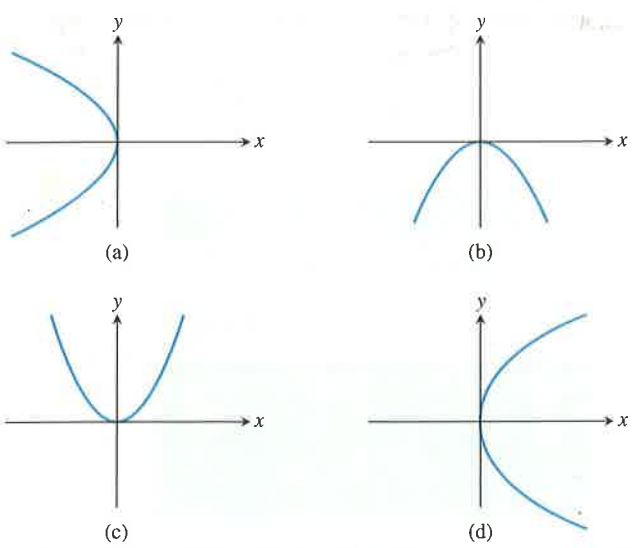
In Exercises 1–6, find the vertex, focus, directrix, and focal width of the parabola.

1. $x^2 = 6y$ **2.** $y^2 = -8x$

3. $(y - 2)^2 = 4(x + 3)$ **4.** $(x + 4)^2 = -6(y + 1)$

5. $3x^2 = -4y$ **6.** $5y^2 = 16x$

In Exercises 7–10, match the graph with its equation.



7. $x^2 = 3y$ **8.** $x^2 = -4y$

9. $y^2 = -5x$ **10.** $y^2 = 10x$

In Exercises 11–30, find an equation in standard form for the parabola that satisfies the given conditions.

11. Vertex $(0, 0)$, focus $(-3, 0)$

12. Vertex $(0, 0)$, focus $(0, 2)$

13. Vertex $(0, 0)$, directrix $y = 4$

14. Vertex $(0, 0)$, directrix $x = -2$

15. Focus $(0, 5)$, directrix $y = -5$

16. Focus $(-4, 0)$, directrix $x = 4$

17. Vertex $(0, 0)$, opens to the right, focal width $= 8$

18. Vertex $(0, 0)$, opens to the left, focal width $= 12$

19. Vertex $(0, 0)$, opens downward, focal width $= 6$

20. Vertex $(0, 0)$, opens upward, focal width $= 3$

21. Focus $(-2, -4)$, vertex $(-4, -4)$

22. Focus $(-5, 3)$, vertex $(-5, 6)$

23. Focus $(3, 4)$, directrix $y = 1$

24. Focus $(2, -3)$, directrix $x = 5$

25. Vertex $(4, 3)$, directrix $x = 6$

26. Vertex $(3, 5)$, directrix $y = 7$

27. Vertex $(2, -1)$, opens upward, focal width $= 16$

28. Vertex $(-3, 3)$, opens downward, focal width $= 20$

29. Vertex $(-1, -4)$, opens to the left, focal width $= 10$

30. Vertex $(2, 3)$, opens to the right, focal width $= 5$

In Exercises 31–36, sketch the graph of the parabola by hand.

31. $y^2 = -4x$ **32.** $x^2 = 8y$

33. $(x + 4)^2 = -12(y + 1)$ **34.** $(y + 2)^2 = -16(x + 3)$

35. $(y - 1)^2 = 8(x + 3)$ **36.** $(x - 5)^2 = 20(y + 2)$

In Exercises 37–48, graph the parabola using a function grapher.

37. $y = 4x^2$

38. $y = -\frac{1}{6}x^2$

39. $x = -8y^2$

40. $x = 2y^2$

41. $12(y + 1) = (x - 3)^2$

42. $6(y - 3) = (x + 1)^2$

43. $2 - y = 16(x - 3)^2$

44. $(x + 4)^2 = -6(y - 1)$

45. $(y + 3)^2 = 12(x - 2)$

46. $(y - 1)^2 = -4(x + 5)$

47. $(y + 2)^2 = -8(x + 1)$

48. $(y - 6)^2 = 16(x - 4)$

In Exercises 49–52, prove that the graph of the equation is a parabola, and find its vertex, focus, and directrix.

49. $x^2 + 2x - y + 3 = 0$

50. $3x^2 - 6x - 6y + 10 = 0$

51. $y^2 - 4y - 8x + 20 = 0$

52. $y^2 - 2y + 4x - 12 = 0$

In Exercises 53–56, write an equation for the parabola.

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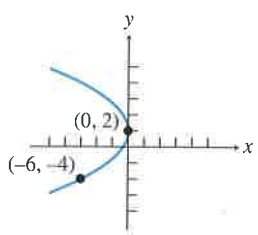

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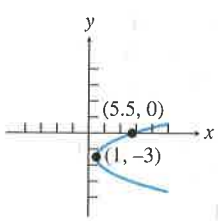

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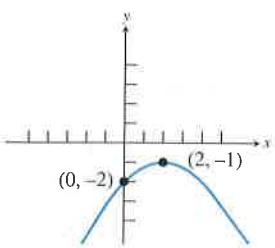

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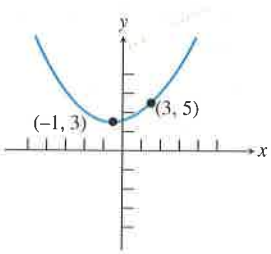
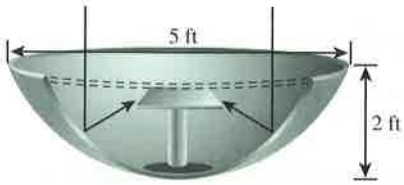


57. Writing to Learn Explain why the derivation of $x^2 = 4py$ is valid regardless of whether $p > 0$ or $p < 0$.

58. Writing to Learn Prove that an equation for the parabola with focus $(p, 0)$ and directrix $x = -p$ is $y^2 = 4px$.

59. Designing a Flashlight Mirror The mirror of a flashlight is a paraboloid of revolution. Its diameter is 6 cm and its depth is 2 cm. How far from the vertex should the filament of the light bulb be placed for the flashlight to have its beam run parallel to the axis of its mirror?

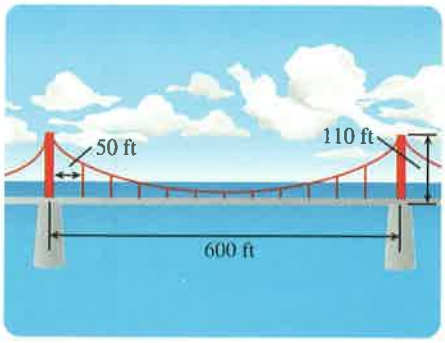
60. Designing a Satellite Dish The reflector of a television satellite dish is a paraboloid of revolution with diameter 5 ft and a depth of 2 ft. How far from the vertex should the receiving antenna be placed?



61. Parabolic Microphones Sports Channel uses a parabolic microphone to capture all the sounds from golf tournaments throughout a season. If one of its microphones has a parabolic surface generated by the parabola $10y = x^2$, locate the focus (the electronic receiver) of the parabola.

62. Parabolic Headlights Stein Glass, Inc., makes parabolic headlights for a variety of automobiles. If one of its headlights has a parabolic surface generated by the parabola $x^2 = 12y$, where should its light bulb be placed?

63. Group Activity Designing a Suspension Bridge The main cables of a suspension bridge uniformly distribute the weight of the bridge when in the form of a parabola. The main cables of a particular bridge are attached to towers that are 600 ft apart. The cables are attached to the towers at a height of 110 ft above the roadway and are 10 ft above the roadway at their lowest points. If vertical support cables are at 50-ft intervals along the level roadway, what are the lengths of these vertical cables?



64. Group Activity Designing a Bridge Arch Parabolic arches are known to have greater strength than other arches. A bridge with a supporting parabolic arch spans 60 ft with a 30-ft wide road passing underneath the bridge. In order to have a minimum clearance of 16 ft, what is the maximum clearance?

Standardized Test Questions

65. True or False Every point on a parabola is the same distance from its focus and its axis. Justify your answer.

66. True or False The directrix of a parabola is parallel to the parabola's axis. Justify your answer.

In Exercises 67–70, solve the problem without using a calculator.

67. Multiple Choice Which of the following curves is **not** a conic section?

(a) circle

(b) ellipse

(c) hyperbola

(d) oval

(e) parabola

68. Multiple Choice Which point do all conics of the form $x^2 = 4py$ have in common?

(a) $(1, 1)$

(b) $(1, 0)$

(c) $(0, 1)$

(d) $(0, 0)$

(e) $(-1, -1)$

69. Multiple Choice The focus of $y^2 = 12x$ is

(a) $(3, 3)$.

(b) $(3, 0)$.

(c) $(0, 3)$.

(d) $(0, 0)$.

(e) $(-3, -3)$.

70. Multiple Choice The vertex of $(y - 3)^2 = -8(x + 2)$ is

(a) $(3, -2)$.

(b) $(-3, -2)$.

(c) $(-3, 2)$.

(d) $(-2, 3)$.

(e) $(-2, -3)$.

Explorations

71. Dynamically Constructing a Parabola Use a geometry software package, such as *Cabri Geometry II*™, *The Geometer's Sketchpad*®, or similar application on a hand-held device to construct a parabola geometrically from its definition. (See Figure 8.3.)

(a) Start by placing a line l (directrix) and a point F (focus) not on the line in the construction window.

(b) Construct a point A on the directrix, and then the segment AF.

(c) Construct a point P where the perpendicular bisector of AF meets the line perpendicular to l through A.

(d) What curve does P trace out as A moves?

(e) Prove your answer to (d) is correct.

72. Constructing Points of a Parabola Use a geometry software package, such as *Cabri Geometry II*™, *The Geometer's Sketchpad*®, or similar application on a hand-held device, to construct Figure 8.4, associated with Exploration 1.

(a) Start by placing the coordinate axes in the construction window.

(b) Construct the line $y = -1$ as the directrix and the point $(0, 1)$ as the focus.

(c) Construct the horizontal lines and concentric circles shown in Figure 8.4.

(d) Construct the points where these horizontal lines and concentric circles meet.

(e) Prove these points lie on the parabola with directrix $y = -1$ and focus $(0, 1)$.

73. Degenerate Cones and Degenerate Conics Degenerate cones occur when the generator and axis of the cone are parallel or perpendicular.

(a) Draw a sketch and describe the "cone" obtained when the generator and axis of the cone are parallel.

(b) Draw sketches and name the types of degenerate conics obtained by intersecting the degenerate cone in (a) with a plane.

(c) Draw a sketch and describe the "cone" obtained when the generator and axis of the cone are perpendicular.

(d) Draw sketches and name the types of degenerate conics obtained by intersecting the degenerate cone in (c) with a plane.

Extending the Ideas

74. Tangent Lines A **tangent line** of a parabola is a line that intersects but does not cross the parabola. Prove that a line tangent to the parabola $x^2 = 4py$ at the point (a, b) crosses the y-axis at $(0, -b)$.

75. Focal Chords A **focal chord** of a parabola is a chord of the parabola that passes through the focus.

(a) Prove that the x-coordinates of the endpoints of a focal chord of $x^2 = 4py$ are $x = 2p(m \pm \sqrt{m^2 + 1})$, where m is the slope of the focal chord.

(b) Using (a), prove the minimum length of a focal chord is the focal width $|4p|$.

76. Latus Rectum The focal chord of a parabola perpendicular to the axis of the parabola is the **lactus rectum** ,Latin for "right chord." Using the results from Exercises 74 and 75, prove:

(a) For a parabola the two endpoints of the latus rectum and the point of intersection of the axis and directrix are the vertices of an isosceles right triangle.

(b) The sides of the right angle of this triangle are tangent to the parabola.

8.2 ELLIPSES

What you'll learn about

- Geometry of an Ellipse
- Translations of Ellipses
- Orbits and Eccentricity
- Reflective Property of an Ellipse

. . . and why

Ellipses are the paths of planets and comets around the Sun, or of moons around planets.

Geometry of an Ellipse

When a plane intersects one nappe of a right circular cylinder and forms a simple closed curve, the curve is an ellipse.

Definition Ellipse

An **ellipse** is the set of all points in a plane whose distances from two fixed points in the plane have a constant sum. The fixed points are the **foci** (plural of focus) of the ellipse. The line through the foci is the **focal axis**. The point on the focal axis midway between the foci is the **center**. The points where the ellipse intersects its axis are the **vertices** of the ellipse. (See Figure 8.11.)

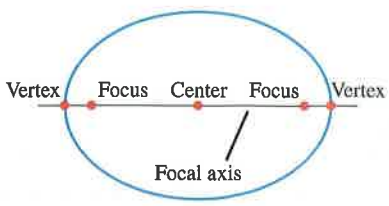
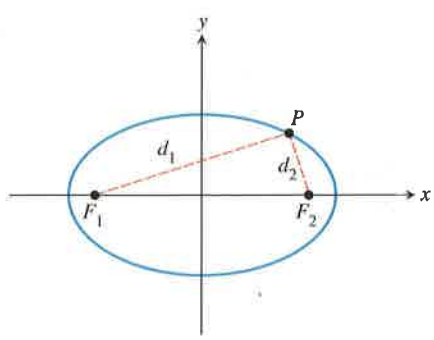


FIGURE 8.11 Key points on the focal axis of an ellipse.



$d_1 + d_2 = \text{constant}$

FIGURE 8.12 *Structure of an Ellipse.* The sum of the distances from the foci to each point on the ellipse is a constant.

Figure 8.12 shows a point $P(x, y)$ of an ellipse. Each of the fixed points, F_1 and F_2, is a focus of the ellipse, and the distances whose sum is constant are

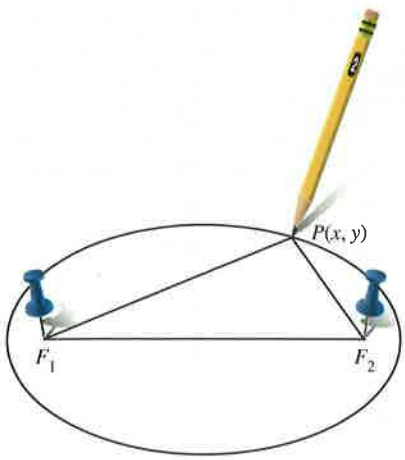


FIGURE 8.13 How to draw an ellipse.

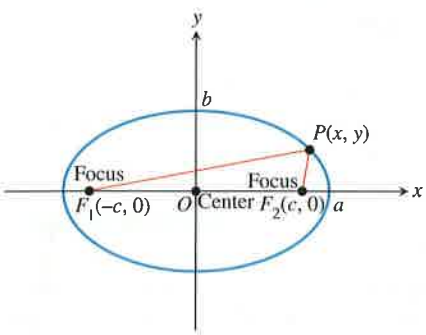


FIGURE 8.14 The ellipse defined by $PF_1 + PF_2 = 2a$ is the graph of the equation $x^2/a^2 + y^2/b^2 = 1$, where $b^2 = a^2 - c^2$.

d_1 and d_2. We can construct an ellipse using a pencil, a loop of string, and two pushpins. Put the loop around the two pins placed at F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace out the ellipse (Figure 8.13).

We now use the definition to derive an equation for an ellipse. For some constants a and c with $a > c \geq 0$, let $F_1(-c, 0)$ and $F_2(c, 0)$ be the foci (Figure 8.14). Then an ellipse is defined by the set of points $P(x, y)$ such that

$$PF_1 + PF_2 = 2a.$$

Using the distance formula, the equation becomes

$$\sqrt{(x+c)^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a.$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \qquad \text{Square.}$$

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx \qquad \text{Simplify.}$$

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2 \qquad \text{Square.}$$

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2) \qquad \text{Simplify.}$$

Letting $b^2 = a^2 - c^2$, we have

$$b^2x^2 + a^2y^2 = a^2b^2$$

which is usually written as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Because these steps can be reversed, a point $P(x, y)$ satisfies this last equation if and only if the point lies on the ellipse defined by $PF_1 + PF_2 = 2a$, provided that $a > c \geq 0$ and $b^2 = a^2 - c^2$. The *Pythagorean relation* $b^2 = a^2 - c^2$ can be written many ways, including $c^2 = a^2 - b^2$ and $a^2 = b^2 + c^2$.

The equation $x^2/a^2 + y^2/b^2 = 1$ is the **standard form** of the equation of an ellipse centered at the origin with the x-axis as its focal axis. An ellipse centered at the origin with the y-axis as its focal axis is the *inverse* of $x^2/a^2 + y^2/b^2 = 1$, and thus has an equation of the form

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

As with circles and parabolas, a line segment with endpoints on an ellipse is a **chord** of the ellipse. The chord lying on the focal axis is the **major axis** of the ellipse. The chord through the center perpendicular to the focal axis is the **minor axis** of the ellipse. The length of the major axis is $2a$, and of the minor axis is $2b$. The number a is the **semimajor axis**, and b is the **semiminor axis**.

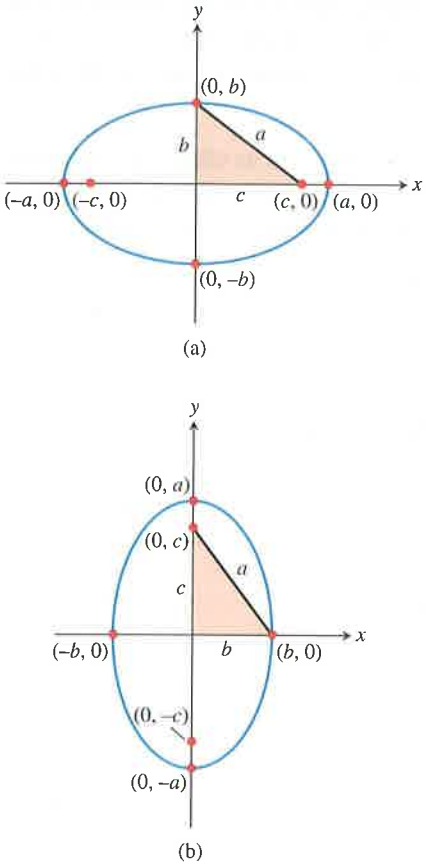


FIGURE 8.15 Ellipses centered at the origin with foci on (a) the x-axis and (b) the y-axis. In each case, a right triangle illustrating the Pythagorean relation is shown.

Ellipses with Center (0, 0)

• **Standard equation**	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
• **Focal axis**	x-axis	y-axis
• **Foci**	$(\pm c, 0)$	$(0, \pm c)$
• **Vertices**	$(\pm a, 0)$	$(0, \pm a)$
• **Semimajor axis**	a	a
• **Semiminor axis**	b	b
• **Pythagorean relation**	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

See Figure 8.15.

EXAMPLE 1 Finding the vertices and foci of an ellipse

Find the vertices and the foci of the ellipse $4x^2 + 9y^2 = 36$.

SOLUTION Dividing both sides of the equation by 36 yields the standard form $x^2/9 + y^2/4 = 1$. Because the larger number is the denominator of x^2, the focal axis is the x-axis. So $a^2 = 9$, $b^2 = 4$, and $c^2 = a^2 - b^2 = 9 - 4 = 5$. Thus the vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{5}, 0)$.

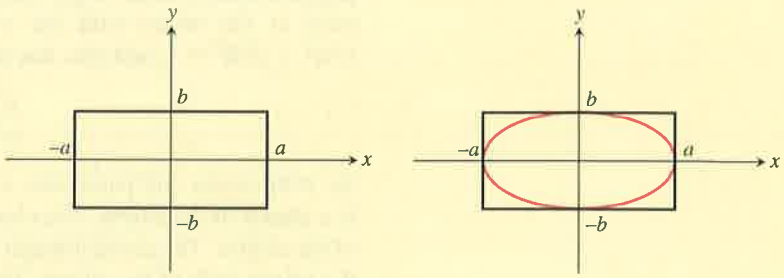
Now try Exercise 1.

An ellipse centered at the origin with its focal axis on a coordinate axis is symmetric with respect to the origin and both coordinate axes. Such an ellipse can be sketched by first drawing a *guiding rectangle* centered at the origin with sides parallel to the coordinate axes and then sketching the ellipse inside the rectangle, as shown in the Drawing Lesson.

Drawing Lesson

How to Sketch the Ellipse $x^2/a^2 + y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$ and complete the rectangle they determine.
2. Inscribe an ellipse that is tangent to the rectangle at $(\pm a, 0)$ and $(0, \pm b)$.



If we wish to graph an ellipse using a function grapher, we need to solve the equation of the ellipse for y, as illustrated in Example 2.

EXAMPLE 2 Finding an equation and graphing an ellipse

Find an equation of the ellipse with foci $(0, -3)$ and $(0, 3)$ whose minor axis has length 4. Sketch the ellipse and support your sketch with a grapher.

SOLUTION The center is $(0, 0)$. The foci are on the y-axis with $c = 3$. The semiminor axis is $b = 4/2 = 2$. Using $a^2 = b^2 + c^2$, we have $a^2 = 2^2 + 3^2 = 13$. So the standard form of the equation for the ellipse is

$$\frac{y^2}{13} + \frac{x^2}{4} = 1.$$

Using $a = \sqrt{13} \approx 3.61$ and $b = 2$, we can sketch a guiding rectangle and then the ellipse itself, as explained in the Drawing Lesson. (Try doing this.) To graph the ellipse using a function grapher, we solve for y in terms of x.

$$\frac{y^2}{13} = 1 - \frac{x^2}{4}$$

$$y^2 = 13(1 - x^2/4)$$

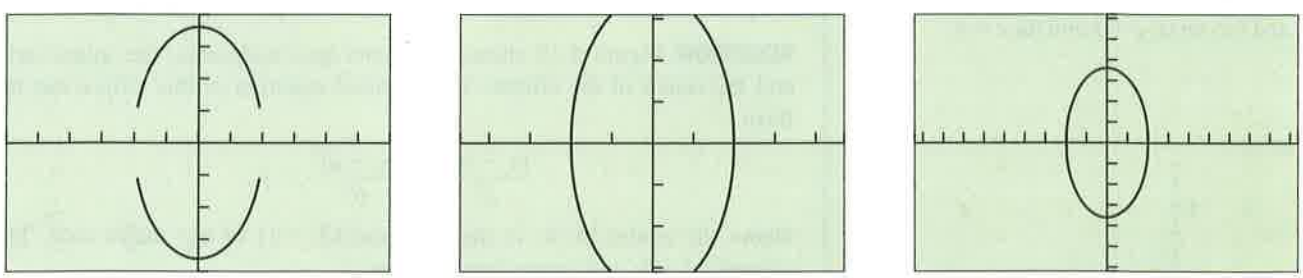
$$y = \pm\sqrt{13(1 - x^2/4)}$$

Figure 8.16 shows three views of the graphs of

$$\text{Y1} = \sqrt{13(1 - x^2/4)} \quad \text{and} \quad \text{Y2} = -\sqrt{13(1 - x^2/4)}.$$

We must select the viewing window carefully to avoid grapher failure.

Now try Exercise 17.



[−6, 6] by [−4, 4] (a) [−4.7, 4.7] by [−3.1, 3.1] (b) [−9.4, 9.4] by [−6.2, 6.2] (c)

FIGURE 8.16 Three views of the ellipse $y^2/13 + x^2/4 = 1$. All of the windows are square or approximately square viewing windows so we can see the true shape. Notice that the gaps between the two function branches do not show when the grapher window includes columns of pixels whose x-coordinates are ± 2 as in (b) and (c). (Example 2)

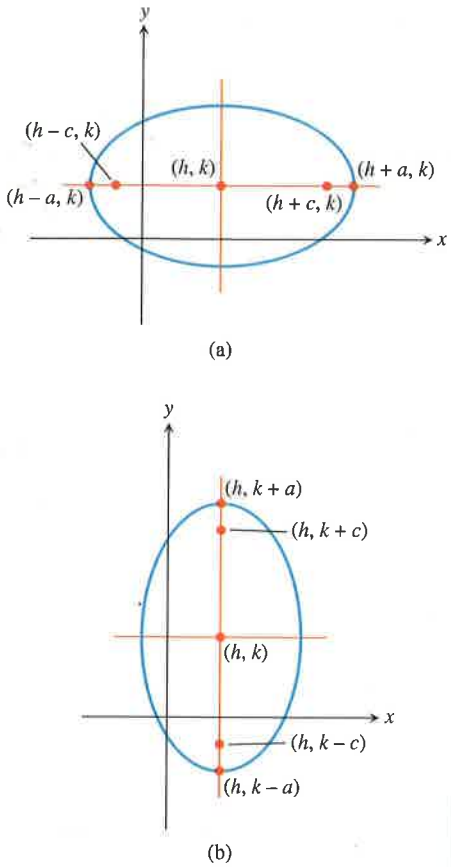


FIGURE 8.17 Ellipses with center (h, k) and foci on (a) $y = k$ and (b) $x = h$.

Translations of Ellipses

When an ellipse with center (0, 0) is translated horizontally by h units and vertically by k units, the center of the ellipse moves from (0, 0) to (h, k), as shown in Figure 8.17. Such a translation does not change the length of the major or minor axis or the Pythagorean relation.

Ellipses with Center (h, k)

• **Standard equation**	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
• **Focal axis**	$y = k$	$x = h$
• **Foci**	$(h \pm c, k)$	$(h, k \pm c)$
• **Vertices**	$(h \pm a, k)$	$(h, k \pm a)$
• **Semimajor axis**	a	a
• **Semiminor axis**	b	b
• **Pythagorean relation**	$a^2 = b^2 + c^2$	$a^2 = b^2 + c^2$

See Figure 8.17.

EXAMPLE 3 Finding an equation of an ellipse

Find the standard form of the equation for the ellipse whose major axis has endpoints $(-2, -1)$ and $(8, -1)$, and whose minor axis has length 8.

SOLUTION Figure 8.18 shows the major-axis endpoints, the minor axis, and the center of the ellipse. The standard equation of this ellipse has the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

where the center (h, k) is the midpoint $(3, -1)$ of the major axis. The semimajor axis and semiminor axis are

$$a = \frac{8-(-2)}{2} = 5 \quad \text{and} \quad b = \frac{8}{2} = 4.$$

So the equation we seek is

$$\frac{(x-3)^2}{5^2} + \frac{(y-(-1))^2}{4^2} = 1,$$

$$\frac{(x-3)^2}{25} + \frac{(y+1)^2}{16} = 1.$$

Now try Exercise 31.

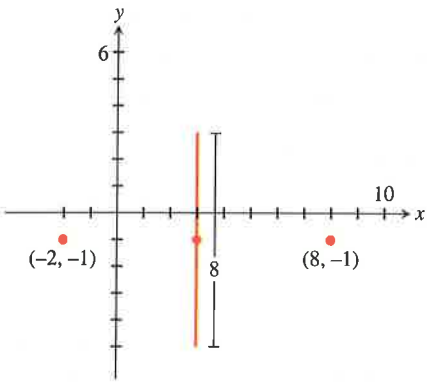
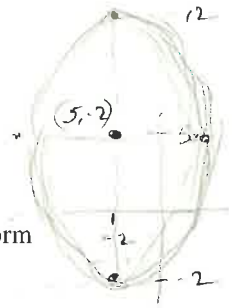


FIGURE 8.18 Given information for Example 3.



EXAMPLE 4 Locating key points of an ellipse

Find the center, vertices, and foci of the ellipse

$$\frac{(x+2)^2}{9} + \frac{(y-5)^2}{49} = 1.$$

SOLUTION The standard equation of this ellipse has the form

$$\frac{(y-5)^2}{49} + \frac{(x+2)^2}{9} = 1.$$

The center (h, k) is $(-2, 5)$. Because the semimajor axis $a = \sqrt{49} = 7$, the vertices $(h, k \pm a)$ are

$$(h, k + a) = (-2, 5 + 7) = (-2, 12) \text{ and}$$

$$(h, k - a) = (-2, 5 - 7) = (-2, -2).$$

Because $c = \sqrt{a^2 - b^2} = \sqrt{49 - 9} = \sqrt{40}$, the foci $(h, k \pm c)$ are $(-2, 5 \pm \sqrt{40})$, or approximately $(-2, 11.32)$ and $(-2, -1.32)$.

Now try Exercise 37.

With the information found about the ellipse in Example 4 and knowing that its semiminor axis $b = \sqrt{9} = 3$, we could easily sketch the ellipse. Obtaining an accurate graph of the ellipse using a function grapher is another matter. Generally, the best way to graph an ellipse using a graphing utility is to use parametric equations.

EXPLORATION 1 Graphing an Ellipse Using Its Parametric Equations

1. Use the Pythagorean trigonometry identity $\cos^2 t + \sin^2 t = 1$ to prove that the parameterization $x = -2 + 3\cos t$, $y = 5 + 7\sin t$, $0 \le t \le 2\pi$ will produce a graph of the ellipse $(x + 2)^2/9 + (y - 5)^2/49 = 1$.
2. Graph $x = -2 + 3\cos t$, $y = 5 + 7\sin t$, $0 \le t \le 2\pi$ in a square viewing window to support part 1 graphically.
3. Create parameterizations for the ellipses in Examples 1, 2, and 3.
4. Graph each of your parameterizations in part 3 and check the features of the obtained graph to see whether they match the expected geometric features of the ellipse. Revise your parameterization and regraph until all features match.
5. Prove that each of your parameterizations is valid.

Orbits and Eccentricity

Kepler's first law of planetary motion, published in 1609, states that the path of a planet's orbit is an ellipse with the Sun at one of the foci. Asteroids, comets, and other bodies that orbit the Sun follow elliptical paths. The closest point to the Sun in such an orbit is the *perihelion*, and the farthest point is the *aphelion* (Figure 8.19). The shape of an ellipse is related to its *eccentricity*.

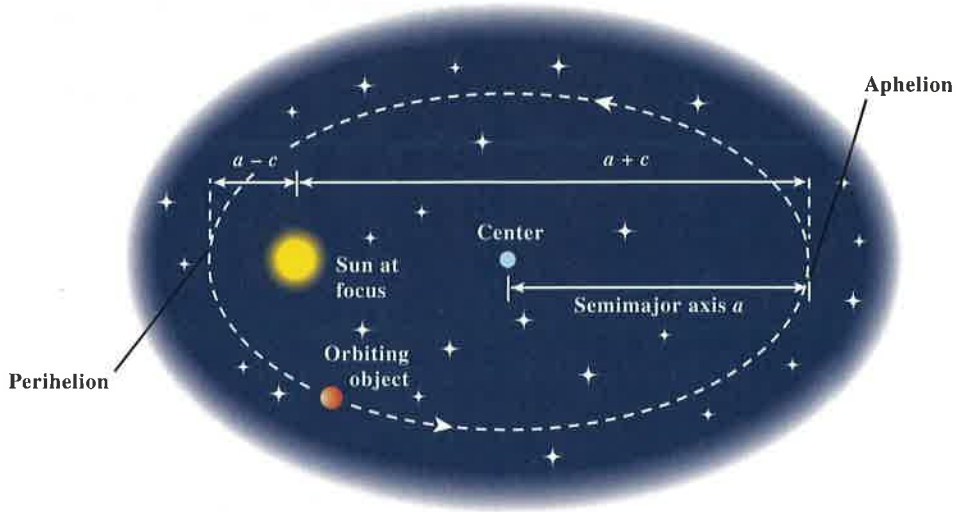


FIGURE 8.19 Many celestial objects have elliptical orbits around the Sun.

A NEW *e*

Try not to confuse the eccentricity e with the natural base e used in exponential and logarithmic functions. The context should clarify which meaning is intended.

Definition Eccentricity of an Ellipse

The **eccentricity** of an ellipse is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a},$$

where a is the semimajor axis, b is the semiminor axis, and c is the distance from the center of the ellipse to either focus.

The eccentricity is the ratio of c to a. The eccentricity e of any ellipse is between 0 and 1, or more precisely, $0 \le e < 1$. (Why?) The larger c is compared to a, the more off-center the foci are. Ellipses with highly off-center foci are elongated and have eccentricities close to 1; for example, the orbit of Halley's comet has eccentricity $e \approx 0.97$. Ellipses with foci near the center are almost circular and have eccentricities close to 0; for instance, Venus's orbit has an eccentricity of 0.0068.

What happens when the eccentricity $e = 0$? In an ellipse, because a is positive, $e = c/a = 0$ implies that $c = 0$ and thus $a = b$. In this case, the ellipse *degenerates* into a circle. Because the ellipse is a circle when $a = b$, it is customary to denote this common value as r and call it the radius of the circle.

Surprising things happen when an ellipse is nearly but not quite a circle, as in the orbit of the planet Earth.

EXAMPLE 5 Analyzing the Earth's orbit

The Earth's orbit has a semimajor axis $a \approx 149.598$ Gm (gigameters) and an eccentricity of $e \approx 0.0167$. Calculate and interpret b and c.

SOLUTION Because $e = c/a$, $c = ea \approx 0.0167 \times 149.598 = 2.4982866$ and

$$b = \sqrt{a^2 - c^2} \approx \sqrt{149.598^2 - 2.4982866^2} \approx 149.577.$$

The semiminor axis $b \approx 149.577$ Gm is only 0.014% shorter than the semimajor axis $a \approx 149.598$ Gm. The aphelion distance of the Earth from the Sun is $a + c \approx 149.598 + 2.498 = 152.096$ Gm, and the perihelion distance is $a - c \approx 149.598 - 2.498 = 147.100$ Gm.

Thus the Earth's orbit is nearly a perfect circle, but the distance between the center of the Sun at one focus and the center of Earth's orbit is $c \approx 2.498$ Gm, more than 2 orders of magnitude greater than $a - b$. The eccentricity as a percentage is 1.67%; this measures how far *off-center* the Sun is.

Now try Exercise 53.

EXPLORATION 2 Constructing Ellipses to Understand Eccentricity

Each group will need a pencil, a centimeter ruler, scissors, some string, several sheets of unlined paper, two pushpins, and a foam board or other appropriate backing material.

1. Make a closed *loop* of string that is 20 cm in circumference.
2. Place a sheet of unlined paper on the backing material, and carefully place the two pushpins 2 cm apart near the center of the paper. Construct an ellipse using the loop of string and a pencil as shown in Figure 8.13. Measure and record the resulting values of a, b, and c for the ellipse, and compute the ratios $e = c/a$ and b/a for the ellipse.
3. On separate sheets of paper repeat step 2 three more times, placing the pushpins 4, 6, and 8 cm apart. Record the values of a, b, c and the ratios e and b/a for each ellipse.
4. Write your observations about the ratio b/a as the eccentricity ratio e increases. Which of these two ratios measures the shape of the ellipse? Which measures how off-center the foci are?
5. Plot the ordered pairs $(e, b/a)$, determine a formula for the ratio b/a as a function of the eccentricity e, and overlay this function's graph on the scatter plot.

Reflective Property of an Ellipse

Because of their shape, ellipses are used to make reflectors of sound, light, and other waves. If we rotate an ellipse in three-dimensional space about its focal axis, the ellipse sweeps out an **ellipsoid of revolution**. If we place a signal source at one focus of a reflective ellipsoid, the signal reflects off the elliptical surface to the other focus, as illustrated in Figure 8.20. This property is used to make mirrors for optical equipment and to study aircraft noise in wind tunnels.

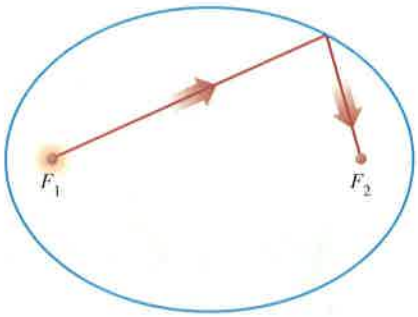


FIGURE 8.20 The reflective property of an ellipse.

In architecture, ceilings in the shape of an ellipsoid are used to create *whispering galleries*. A person whispering at one focus can be heard across the room by a person at the other focus. An ellipsoid is part of the design of the Texas state capitol; a hand clap made in the center of the main vestibule (at one focus of the ellipsoid) bounces off the inner elliptical dome, passes through the other focus, bounces off the dome a second time, and returns to the person as a distinct echo.

Ellipsoids are used in health care to avoid surgery in the treatment of kidney stones. An elliptical *lithotripter* emits underwater ultrahigh-frequency (UHF) shock waves from one focus, with the patient's kidney carefully positioned at the other focus (Figure 8.21).

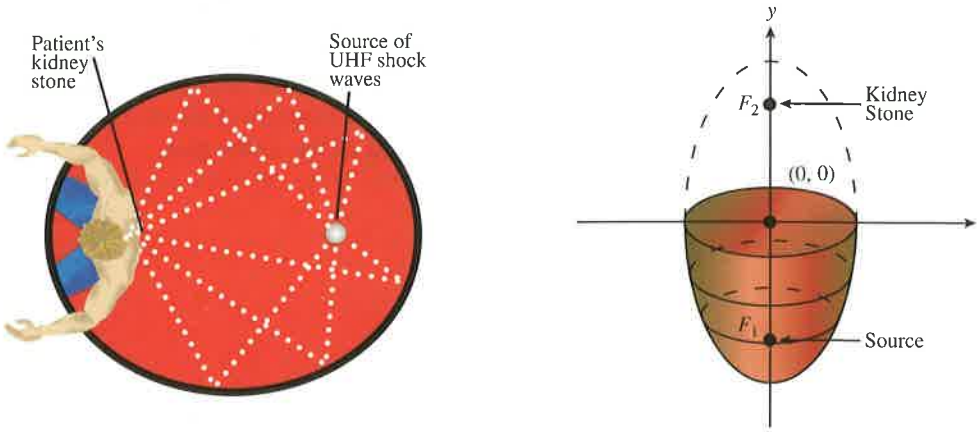


FIGURE 8.21 How a lithotripter breaks up kidney stones.

EXAMPLE 6 Focusing a lithotripter

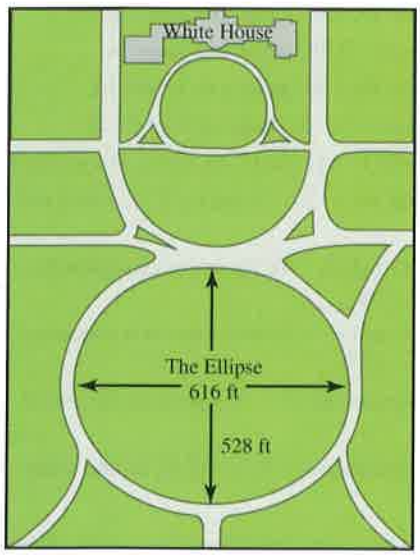
The ellipse used to generate the ellipsoid of a lithotripter has a major axis of 12 ft and a minor axis of 5 ft. How far from the center are the foci?

SOLUTION From the given information, we know $a = 12/2 = 6$ and $b = 5/2 = 2.5$. So

$$c = \sqrt{a^2 - b^2} \approx \sqrt{6^2 - 2.5^2} \approx 5.4544.$$

So the foci are about 5 ft 5.5 inches from the center of the lithotripter.

Now try Exercise 59.



PROBLEM: If the Ellipse at the White House is 616 ft long and 528 ft wide, what is its equation?

SOLUTION: For simplicity's sake, we model the Ellipse as centered at (0, 0) with the x-axis as its focal axis. Because the Ellipse is 616 ft long, $a = 616/2 = 308$, and because the Ellipse is 528 ft wide, $b = 528/2 = 264$. Using $x^2/a^2 + y^2/b^2 = 1$, we obtain

$$\frac{x^2}{308^2} + \frac{y^2}{264^2} = 1,$$

$$\frac{x^2}{94{,}864} + \frac{y^2}{69{,}696} = 1.$$

Other models are possible.

QUICK REVIEW 8.2

(For help, go to Sections P.2 and P.5.)

In Exercises 1 and 2, find the distance between the given points

1. $(-3, -2)$ and $(2, 4)$

2. $(-3, -4)$ and (a, b)

In Exercises 3 and 4, solve for y in terms of x.

3. $\frac{y^2}{9} + \frac{x^2}{4} = 1$

4. $\frac{x^2}{36} + \frac{y^2}{25} = 1$

In Exercises 5–8, solve for x algebraically.

5. $\sqrt{3x + 12} + \sqrt{3x - 8} = 10$

6. $\sqrt{6x + 12} - \sqrt{4x + 9} = 1$

7. $\sqrt{6x^2 + 12} + \sqrt{6x^2 + 1} = 11$

8. $\sqrt{2x^2 + 8} + \sqrt{3x^2 + 4} = 8$

In Exercises 9 and 10, find exact solutions by completing the square.

9. $2x^2 - 6x - 3 = 0$

10. $2x^2 + 4x - 5 = 0$

SECTION 8.2 EXERCISES

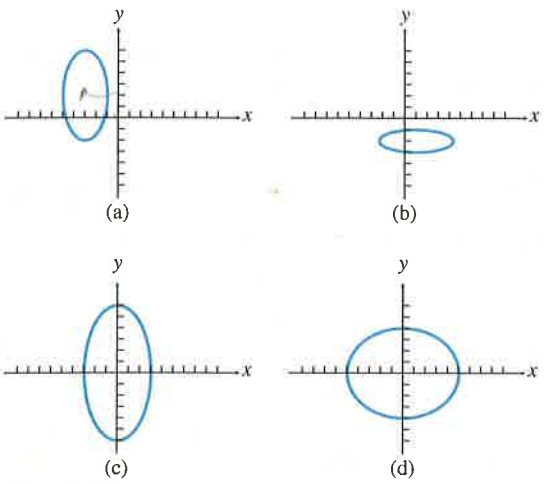
In Exercises 1–6, find the vertices and foci of the ellipse.

1. $\frac{x^2}{16} + \frac{y^2}{7} = 1$ **2.** $\frac{y^2}{25} + \frac{x^2}{21} = 1$

3. $\frac{y^2}{36} + \frac{x^2}{27} = 1$ **4.** $\frac{x^2}{11} + \frac{y^2}{7} = 1$

5. $3x^2 + 4y^2 = 12$ **6.** $9x^2 + 4y^2 = 36$

In Exercises 7–10, match the graph with its equation, given that the ticks on all axes are 1 unit apart.



7. $\frac{x^2}{25} + \frac{y^2}{16} = 1$ **8.** $\frac{y^2}{36} + \frac{x^2}{9} = 1$

9. $\frac{(y-2)^2}{16} + \frac{(x+3)^2}{4} = 1$ **10.** $\frac{(x-1)^2}{11} + (y+2)^2 = 1$

In Exercises 11–16, sketch the graph of the ellipse by hand.

11. $\frac{x^2}{64} + \frac{y^2}{36} = 1$ **12.** $\frac{x^2}{81} + \frac{y^2}{25} = 1$

13. $\frac{y^2}{9} + \frac{x^2}{4} = 1$ **14.** $\frac{y^2}{49} + \frac{x^2}{25} = 1$

15. $\frac{(x+3)^2}{16} + \frac{(y-1)^2}{4} = 1$ **16.** $\frac{(x-1)^2}{2} + \frac{(y+3)^2}{4} = 1$

In Exercises 17–20, graph the ellipse using a function grapher.

17. $\frac{x^2}{36} + \frac{y^2}{16} = 1$ **18.** $\frac{y^2}{64} + \frac{x^2}{16} = 1$

19. $\frac{(x+2)^2}{5} + 2(y-1)^2 = 1$ **20.** $\frac{(x-4)^2}{16} + 16(y+4)^2 = 8$

In Exercises 21–36, find an equation in standard form for the ellipse that satisfies the given conditions.

21. Major axis length 6 on y-axis, minor axis length 4

22. Major axis length 14 on x-axis, minor axis length 10

23. Foci $(\pm 2, 0)$, major axis length 10

24. Foci $(0, \pm 3)$, major axis length 10

25. Endpoints of axes are $(\pm 4, 0)$ and $(0, \pm 5)$

26. Endpoints of axes are $(\pm 7, 0)$ and $(0, \pm 4)$

27. Major axis endpoints $(0, \pm 6)$, minor axis length 8

28. Major axis endpoints $(\pm 5, 0)$, minor axis length 4

29. Minor axis endpoints $(0, \pm 4)$, major axis length 10

30. Minor axis endpoints $(\pm 12, 0)$, major axis length 26

31. Major axis endpoints $(1, -4)$ and $(1, 8)$, minor axis length 8

32. Major axis endpoints are $(-2, -3)$ and $(-2, 7)$, minor axis length 4

33. The foci are $(1, -4)$ and $(5, -4)$; the major axis endpoints are $(0, -4)$ and $(6, -4)$.

34. The foci are $(-2, 1)$ and $(-2, 5)$; the major axis endpoints are $(-2, -1)$ and $(-2, 7)$.

35. The major axis endpoints are $(3, -7)$ and $(3, 3)$; the minor axis length is 6.

36. The major axis endpoints are $(-5, 2)$ and $(3, 2)$; the minor axis length is 6.

In Exercises 37–40, find the center, vertices, and foci of the ellipse.

37. $\frac{(x+1)^2}{25} + \frac{(y-2)^2}{16} = 1$ **38.** $\frac{(x-3)^2}{11} + \frac{(y-5)^2}{7} = 1$

39. $\frac{(y+3)^2}{81} + \frac{(x-7)^2}{64} = 1$ **40.** $\frac{(y-1)^2}{25} + \frac{(x+2)^2}{16} = 1$

In Exercises 41–44, graph the ellipse using a parametric grapher.

41. $\frac{y^2}{25} + \frac{x^2}{4} = 1$ **42.** $\frac{x^2}{30} + \frac{y^2}{20} = 1$

43. $\frac{(x+3)^2}{12} + \frac{(y-6)^2}{5} = 1$ **44.** $\frac{(y+1)^2}{15} + \frac{(x-2)^2}{6} = 1$

In Exercises 45–48, prove that the graph of the equation is an ellipse, and find its vertices, foci, and eccentricity.

45. $9x^2 + 4y^2 - 18x + 8y - 23 = 0$

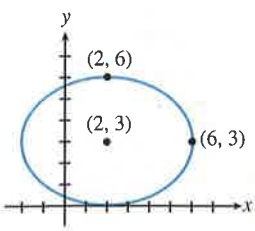
46. $3x^2 + 5y^2 - 12x + 30y + 42 = 0$

47. $9x^2 + 16y^2 + 54x - 32y - 47 = 0$

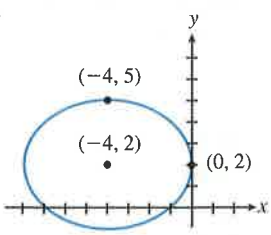
48. $4x^2 + y^2 - 32x + 16y + 124 = 0$

In Exercises 49 and 50, write an equation for the ellipse.

49.



50.



51. Writing to Learn Prove that an equation for the ellipse with center (0, 0), foci $(0, \pm c)$, and semimajor axis $a > c \geq 0$ is $y^2/a^2 + x^2/b^2 = 1$, where $b^2 = a^2 - c^2$.
[*Hint*: Refer to derivation at the beginning of the section.]

52. Dancing Planets Writing to Learn Using the data in Table 8.1, prove that the planet with the most eccentric orbit sometimes is closer to the Sun than the planet with the least eccentric orbit.

TABLE 8.1 SEMIMAJOR AXES AND ECCENTRICITIES OF THE PLANETS

Planet	Semimajor Axis (Gm)	Eccentricity
Mercury	57.9	0.2056
Venus	108.2	0.0068
Earth	149.6	0.0167
Mars	227.9	0.0934
Jupiter	778.3	0.0485
Saturn	1427	0.0560
Uranus	2869	0.0461
Neptune	4497	0.0050
Pluto	5900	0.2484

Source: Shupe, et al., National Geographic Atlas of the World (rev. 6th ed.). Washington, DC: National Geographic Society, 1992, plate 116, and other sources.

53. The Moon's Orbit The Moon's apogee (farthest distance from the Earth) is 252,710 miles, and perigee (closest distance to the Earth) is 221,463 miles. Assuming the Moon's orbit of the Earth is elliptical with the Earth at one focus, calculate and interpret a, b, c, and e.

54. Hot Mercury Given that the diameter of the Sun is about 1.392 Gm, how close does Mercury get to the Sun's surface?

55. Saturn Find the perihelion and aphelion distances of Saturn.

56. Venus and Mars Write equations for the orbits of Venus and Mars in the form $x^2/a^2 + y^2/b^2 = 1$.

57. Sungrazers One comet group, known as the sungrazers, passes within a Sun's diameter (1.392 Gm) of the solar surface. What can you conclude about $a - c$ for orbits of the sungrazers?

58. Halley's Comet The orbit of Halley's comet is 36.18 AU long and 9.12 AU wide. What is its eccentricity?

59. Lithotripter For an ellipse that generates the ellipsoid of a lithotripter, the major axis has endpoints $(-8, 0)$ and $(8, 0)$. One endpoint of the minor axis is $(0, 3.5)$. Find the coordinates of the foci.

60. Lithotripter (Refer to Figure 8.21.) A lithotripter's shape is formed by rotating the portion of an ellipse below its minor axis about its major axis. If the length of the major axis is 26 in. and the length of the minor axis is 10 in., where should the shock-wave source and the patient be placed for maximum effect?

Group Activities In Exercises 61 and 62, solve the system of equations algebraically and support your answer graphically.

61. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
$x^2 + y^2 = 4$

62. $\dfrac{x^2}{9} + y^2 = 1$
$x - 3y = -3$

63. Group Activity Consider the system of equations

$$x^2 + 4y^2 = 4$$
$$y = 2x^2 - 3$$

(a) Solve the system graphically.

(b) If you have access to a grapher that also does symbolic algebra, use it to find the exact solutions to the system.

64. Writing to Learn Look up the adjective *eccentric* in a dictionary and read its various definitions. Notice that the word is derived from *ex-centric*, meaning "out-of-center" or "off-center." Explain how this is related to the word's everyday meanings as well as its mathematical meaning for ellipses.

Standardized Test Questions

65. True or False The distance from a focus of an ellipse to the closer vertex is $a(1+e)$, where a is the semimajor axis and e is the eccentricity. Justify your answer.

66. True or False The distance from a focus of an ellipse to either endpoint of the minor axis is half the length of the major axis. Justify your answer.

In Exercises 67–70, you may use a graphing calculator to solve the problem.

67. Multiple Choice One focus of $x^2 + 4y^2 = 4$ is

(a) $(4, 0)$.

(b) $(2, 0)$.

(c) $(\sqrt{3}, 0)$.

(d) $(\sqrt{2}, 0)$.

(e) $(1, 0)$.

68. Multiple Choice The focal axis of $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{16} = 1$ is

(a) $y = 1$.

(b) $y = 2$.

(c) $y = 3$.

(d) $y = 4$.

(e) $y = 5$.

69. Multiple Choice The center of $9x^2 + 4y^2 - 72x - 24y + 144 = 0$ is

(a) (4, 2).

(b) (4, 3).

(c) (4, 4).

(d) (4, 5).

(e) (4, 6).

70. Multiple Choice The perimeter of a triangle with one vertex on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the other two vertices at the foci of the ellipse would be

(a) $a + b$.

(b) $2a + 2b$.

(c) $2a + 2c$.

(d) $2b + 2c$.

(e) $a + b + c$.

Explorations

71. Area and Perimeter The area of an ellipse is $A = \pi ab$, but the perimeter cannot be expressed so simply:

$$P \approx \pi(a+b)\left(3 - \frac{\sqrt{(3a+b)(a+3b)}}{a+b}\right)$$

(a) Prove that, when $a = b = r$, these become the familiar formulas for the area and perimeter (circumference) of a circle.

(b) Find a pair of ellipses such that the one with greater area has smaller perimeter.

72. Writing to Learn Kepler's Laws We have encountered Kepler's First and Third Laws (p. 186). Using a library or the Internet,

(a) Read about Kepler's life, and write in your own words how he came to discover his three laws of planetary motion.

(b) What is Kepler's Second Law? Explain it with both pictures and words.

Extending the Ideas

73. Prove that a nondegenerate graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is an ellipse if $AC > 0$.

74. Writing to Learn The graph of the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 0$$

is considered to be a degenerate ellipse. Describe the graph. How is it like a full-fledged ellipse, and how is it different?

8.3 HYPERBOLAS

What you'll learn about

- Geometry of a Hyperbola
- Translations of Hyperbolas
- Eccentricity and Orbits
- Reflective Property of a Hyperbola
- Long-Range Navigation

. . . and why

The hyperbola is the least known conic section, yet it is used in astronomy, optics, and navigation.

Geometry of a Hyperbola

When a plane intersects both nappes of a right circular cylinder, the intersection is a hyperbola. The definition, features, and derivation for a hyperbola closely resemble those for an ellipse. As you read on, you may find it helpful to compare the nature of the hyperbola with the nature of the ellipse.

Definition Hyperbola

A hyperbola is the set of all points in a plane whose distances from two fixed points in the plane have a constant *difference*. The fixed points are the **foci** of the hyperbola. The line through the foci is the **focal axis**. The point on the focal axis midway between the foci is the **center**. The points where the hyperbola intersects its focal axis are the **vertices** of the hyperbola. (See Figure 8.22.)

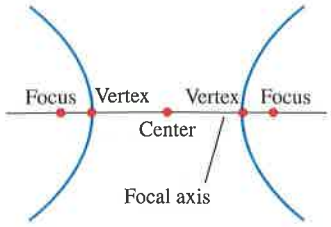


FIGURE 8.22 Key points on the focal axis of a hyperbola.

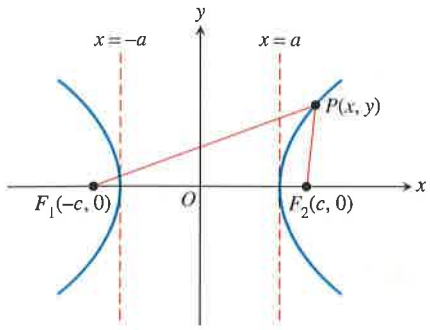


FIGURE 8.23 *Structure of a Hyperbola.* The difference of the distances from the foci to each point on the hyperbola is a constant.

Figure 8.23 shows a hyperbola centered at the origin with its focal axis on the x-axis. The vertices are at $(-a, 0)$ and $(a, 0)$, where a is some positive constant. The fixed points $F_1(-c, 0)$ and $F_2(c, 0)$ are the foci of the hyperbola, with $c > a$.

Notice that the hyperbola has two *branches*. For a point $P(x, y)$ on the right-hand branch, $PF_1 - PF_2 = 2a$. On the left-hand branch, $PF_2 - PF_1 = 2a$. Combining these two equations gives us

$$PF_1 - PF_2 = \pm 2a.$$

Using the distance formula, the equation becomes

$$\sqrt{(x+c)^2 + (y-0)^2} - \sqrt{(x-c)^2 + (y-0)^2} = \pm 2a.$$

$$\sqrt{(x-c)^2 + y^2} = \pm 2a + \sqrt{(x+c)^2 + y^2}$$

$$x^2 - 2cx + c^2 + y^2 = 4a^2 \pm 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2 \qquad \text{Square.}$$

$$\mp a\sqrt{(x+c)^2 + y^2} = a^2 + cx \qquad \text{Simplify.}$$

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2 \qquad \text{Square.}$$

$$(c^2 - a^2)x^2 - a^2y^2 = a^2(c^2 - a^2) \qquad \text{Simplify.}$$

Letting $b^2 = c^2 - a^2$, we have

$$b^2x^2 - a^2y^2 = a^2b^2,$$

which is usually written as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Because these steps can be reversed, a point $P(x, y)$ satisfies this last equation if and only if the point lies on the hyperbola defined by $PF_1 - PF_2 = \pm 2a$, provided that $c > a > 0$ and $b^2 = c^2 - a^2$. The *Pythagorean relation* $b^2 = c^2 - a^2$ can be written many ways, including $a^2 = c^2 - b^2$ and $c^2 = a^2 + b^2$.

The equation $x^2/a^2 - y^2/b^2 = 1$ is the **standard form** of the equation of a hyperbola centered at the origin with the x-axis as its focal axis. A hyperbola centered at the origin with the y-axis as its focal axis is the *inverse relation* of $x^2/a^2 - y^2/b^2 = 1$, and thus has an equation of the form

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

As with other conics, a line segment with endpoints on a hyperbola is a **chord** of the hyperbola. The chord lying on the focal axis connecting the vertices is the **transverse axis** of the hyperbola. The length of the transverse axis is $2a$. The line segment of length $2b$ that is perpendicular to the focal axis and that has the center of the hyperbola as its midpoint is the **conjugate axis** of the hyperbola. The number a is the **semitransverse axis**, and b is the **semiconjugate axis**.

NAMING AXES

The word "transverse" comes from the Latin *trans vertere*: to go across. The transverse axis "goes across" from one vertex to the other. The conjugate axis is the transverse axis for the *conjugate hyperbola*, defined in Exercise 73.

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two *asymptotes*. These asymptotes are slant lines that can be found by replacing the 1 on the right-hand side of the hyperbola's equation by a 0:

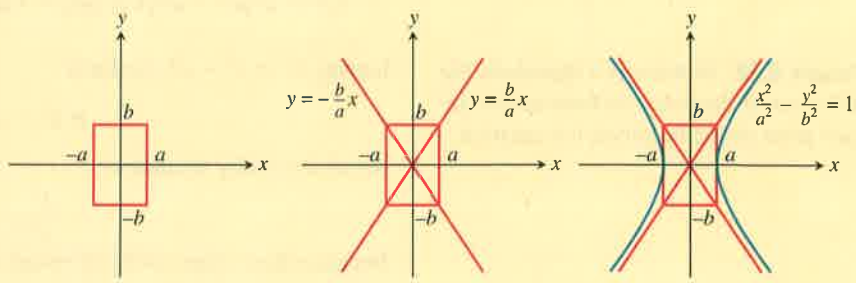
$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{Replace 1 by 0.}} \rightarrow \underbrace{y = \pm\frac{b}{a}x}_{\text{asymptotes}}$$

A hyperbola centered at the origin with its focal axis one of the coordinate axes is symmetric with respect to the origin and both coordinate axes. Such a hyperbola can be sketched by drawing a rectangle centered at the origin with sides parallel to the coordinate axes, followed by drawing the asymptotes through opposite corners of the rectangle, and finally sketching the hyperbola using the central rectangle and asymptotes as guides, as shown in the Drawing Lesson.

Drawing Lesson

How to Sketch the Hyperbola $x^2/a^2 - y^2/b^2 = 1$

1. Sketch line segments at $x = \pm a$ and $y = \pm b$, and complete the rectangle they determine.
2. Sketch the asymptotes by extending the rectangle's diagonals.
3. Use the rectangle and asymptotes to guide your drawing.



Hyperbolas with Center (0, 0)

• **Standard equation**	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
• **Focal axis**	x-axis	y-axis
• **Foci**	$(\pm c, 0)$	$(0, \pm c)$
• **Vertices**	$(\pm a, 0)$	$(0, \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

See Figure 8.24.

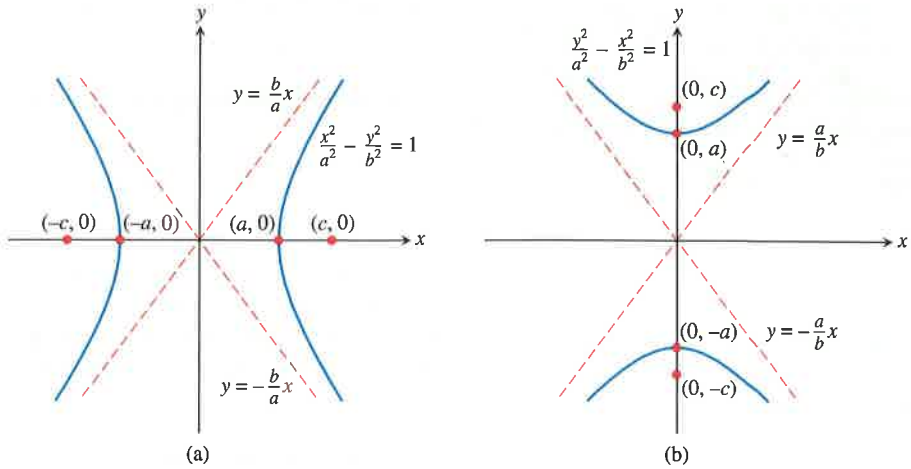


FIGURE 8.24 Hyperbolas centered at the origin with foci on (a) the x-axis and (b) the y-axis.

EXAMPLE 1 Finding the vertices and foci of a hyperbola

Find the vertices and the foci of the hyperbola $4x^2 - 9y^2 = 36$.

SOLUTION Dividing both sides of the equation by 36 yields the standard form $x^2/9 - y^2/4 = 1$. So $a^2 = 9$, $b^2 = 4$, and $c^2 = a^2 + b^2 = 9 + 4 = 13$. Thus the vertices are $(\pm 3, 0)$, and the foci are $(\pm\sqrt{13}, 0)$.

Now try Exercise 1.

If we wish to graph a hyperbola using a function grapher, we need to solve the equation of the hyperbola for y, as illustrated in Example 2.

EXAMPLE 2 Finding an equation and graphing a hyperbola

Find an equation of the hyperbola with foci $(0, -3)$ and $(0, 3)$ whose conjugate axis has length 4. Sketch the hyperbola and its asymptotes, and support your sketch with a grapher.

SOLUTION The center is $(0, 0)$. The foci are on the y-axis with $c = 3$. The semiconjugate axis is $b = 4/2 = 2$. Thus $a^2 = c^2 - b^2 = 3^2 - 2^2 = 5$. So the standard form of the equation for the hyperbola is

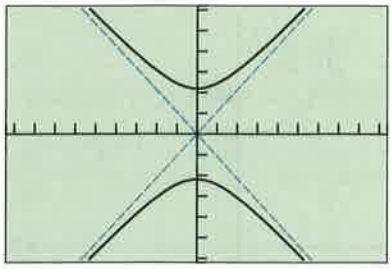
$$\frac{y^2}{5} - \frac{x^2}{4} = 1.$$

Using $a = \sqrt{5} \approx 2.24$ and $b = 2$, we can sketch the central rectangle, the asymptotes, and the hyperbola itself. Try doing this. To graph the hyperbola using a function grapher, we solve for y in terms of x.

$$\frac{y^2}{5} = 1 + \frac{x^2}{4} \qquad \text{Add } \frac{x^2}{4}.$$

$$y^2 = 5(1 + x^2/4) \qquad \text{Multiply by 5.}$$

$$y = \pm\sqrt{5(1 + x^2/4)} \qquad \text{Extract square roots.}$$



[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 8.25 The hyperbola $y^2/5 - x^2/4 = 1$, shown with its asymptotes. (Example 2)

Figure 8.25 shows the graphs of

$$Y1 = \sqrt{5(1 + x^2/4)} \quad \text{and} \quad Y2 = -\sqrt{5(1 + x^2/4)},$$

together with the asymptotes of the hyperbola

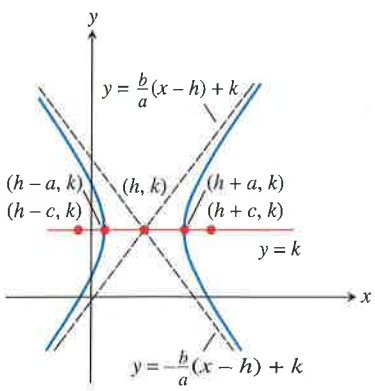
$$Y3 = \frac{\sqrt{5}}{2}x \quad \text{and} \quad Y4 = -\frac{\sqrt{5}}{2}x.$$

Now try Exercise 17.

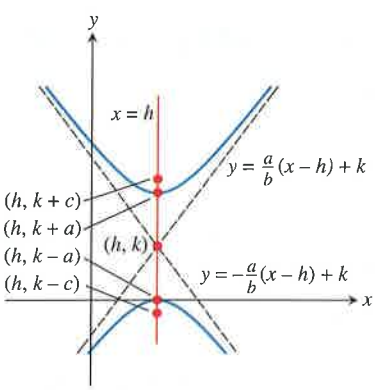
In Example 2, because the hyperbola had a vertical focal axis, selecting a viewing rectangle was easy. When a hyperbola has a horizontal focal axis, we try to select a viewing window to include the two vertices in the plot and thus avoid gaps in the graph of the hyperbola.

Translations of Hyperbolas

When a hyperbola with center (0, 0) is translated horizontally by h units and vertically by k units, the center of the hyperbola moves from (0, 0) to (h, k), as shown in Figure 8.26. Such a translation does not change the length of the transverse or conjugate axis or the Pythagorean relation.



(a)



(b)

FIGURE 8.26 Hyperbolas with center (h, k) and foci on (a) $y = k$ and (b) $x = h$.

Hyperbolas with Center (h, k)

• **Standard equation**	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
• **Focal axis**	$y = k$	$x = h$
• **Foci**	$(h \pm c, k)$	$(h, k \pm c)$
• **Vertices**	$(h \pm a, k)$	$(h, k \pm a)$
• **Semitransverse axis**	a	a
• **Semiconjugate axis**	b	b
• **Pythagorean relation**	$c^2 = a^2 + b^2$	$c^2 = a^2 + b^2$
• **Asymptotes**	$y = \pm\frac{b}{a}(x - h) + k$	$y = \pm\frac{a}{b}(x - h) + k$

See Figure 8.26.

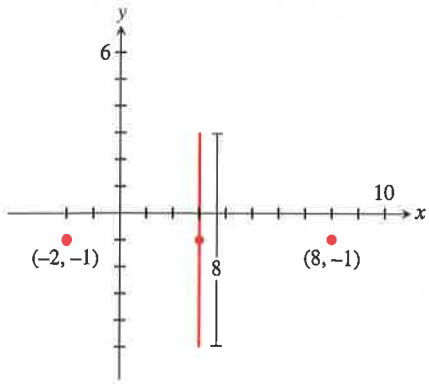


FIGURE 8.27 Given information for Example 3.

EXAMPLE 3 Finding an equation of a hyperbola

Find the standard form of the equation for the hyperbola whose transverse axis has endpoints $(-2, -1)$ and $(8, -1)$, and whose conjugate axis has length 8.

SOLUTION Figure 8.27 shows the transverse-axis endpoints, the conjugate axis, and the center of the hyperbola. The standard equation of this hyperbola has the form

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1,$$

where the center (h, k) is the midpoint $(3, -1)$ of the transverse axis. The semitransverse axis and semiconjugate axis are

$$a = \frac{8-(-2)}{2} = 5 \quad \text{and} \quad b = \frac{8}{2} = 4.$$

So the equation we seek is

$$\frac{(x-3)^2}{5^2} - \frac{(y-(-1))^2}{4^2} = 1,$$

$$\frac{(x-3)^2}{25} - \frac{(y+1)^2}{16} = 1.$$

Now try Exercise 31.

EXAMPLE 4 Locating key points of a hyperbola

Find the center, vertices, and foci of the hyperbola

$$\frac{(x+2)^2}{9} - \frac{(y-5)^2}{49} = 1.$$

SOLUTION The center (h, k) is $(-2, 5)$. Because the semitransverse axis $a = \sqrt{9} = 3$, the vertices are

$$(h+a, k) = (-2+3, 5) = (1, 5) \quad \text{and}$$

$$(h-a, k) = (-2-3, 5) = (-5, 5).$$

Because $c = \sqrt{a^2+b^2} = \sqrt{9+49} = \sqrt{58}$, the foci $(h \pm c, k)$ are $(-2 \pm \sqrt{58}, 5)$, or approximately $(5.62, 5)$ and $(-9.62, 5)$.

Now try Exercise 39.

With the information found about the hyperbola in Example 4 and knowing that its semiconjugate axis $b = \sqrt{49} = 7$, we could easily sketch the hyperbola. Obtaining an accurate graph of the hyperbola using a function grapher is another matter. Often, the best way to graph a hyperbola using a graphing utility is to use parametric equations.

EXPLORATION 1 **Graphing a Hyperbola Using Its Parametric Equations**

1. Use the Pythagorean trigonometry identity $\sec^2 t - \tan^2 t = 1$ to prove that the parameterization $x = -1 + 3/\cos t$, $y = 1 + 2\tan t$ $(0 \le t \le 2\pi)$ will produce a graph of the hyperbola $(x + 1)^2/9 - (y - 1)^2/4 = 1$.
2. Using Dot graphing mode, graph $x = -1 + 3/\cos t$, $y = 1 + 2\tan t$ $(0 \le t \le 2\pi)$ in a square viewing window to support part 1 graphically. Switch to Connected graphing mode, and regraph the equation. What do you observe? Explain.
3. Create parameterizations for the hyperbolas in Examples 1, 2, 3, and 4.
4. Graph each of your parameterizations in part 3 and check the features of the obtained graph to see whether they match the expected geometric features of the hyperbola. If necessary, revise your parameterization and regraph until all features match.
5. Prove that each of your parameterizations is valid.

Eccentricity and Orbits

Definition Eccentricity of a Hyperbola

The **eccentricity** of a hyperbola is

$$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a},$$

where a is the semitransverse axis, b is the semiconjugate axis, and c is the distance from the center to either focus.

For a hyperbola, because $c > a$, the eccentricity $e > 1$. In Section 8.2 we learned that the eccentricity of an ellipse satisfies the inequality $0 \le e < 1$ and that, for $e = 0$, the ellipse is a circle. In Section 8.5 we will generalize the concept of eccentricity to all types of conics and learn that the eccentricity of a parabola is $e = 1$.

Kepler's first law of planetary motion says that a planet's orbit is elliptical with the Sun at one focus. Since 1609, astronomers have generalized Kepler's law; the current theory states: A celestial body that travels within the gravitational field of a much more massive body follows a path that closely approximates a conic section that has the more massive body as a focus. Two bodies that do not differ greatly in mass (such as the Earth and the Moon, or Pluto and its moon Charon) actually revolve around their balance point, or *barycenter*. In theory, a comet can approach the Sun from interstellar space, make a partial loop about the Sun, and then leave the solar system returning to deep space; such a comet follows a path that is one branch of a hyperbola.

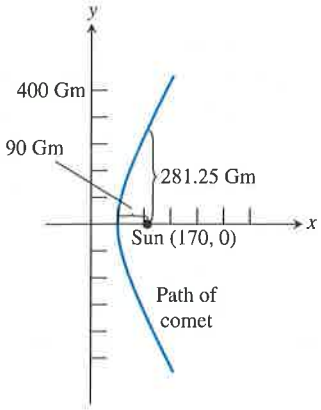


FIGURE 8.28 The graph of one branch of $x^2/6400 - y^2/22{,}500 = 1$. (Example 5)

EXAMPLE 5 Analyzing a comet's orbit

A comet following a hyperbolic path about the Sun has a perihelion distance of 90 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 281.25 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if we coordinatize space so that the hyperbola is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1?$$

SOLUTION The perihelion distance is $c - a = 90$. When $x = c$, $y = \pm b^2/a$ (see Exercise 74). So $b^2/a = 281.25$, or $b^2 = 281.25a$. Because $b^2 = c^2 - a^2$, we have the system

$$c - a = 90 \quad \text{and} \quad c^2 - a^2 = 281.25a,$$

which yields the equation:

$$(a + 90)^2 - a^2 = 281.25a$$

$$a^2 + 180a + 8100 - a^2 = 281.25a$$

$$8100 = 101.25a$$

$$a = 80$$

So $a = 80$ Gm, $b = 150$ Gm, $c = 170$ Gm, and $e = 17/8 = 2.125$. If the comet's path is the branch of the hyperbola with positive x-coordinates, then the Sun is at the focus $(c, 0) = (170, 0)$. See Figure 8.28.

Now try Exercise 55.

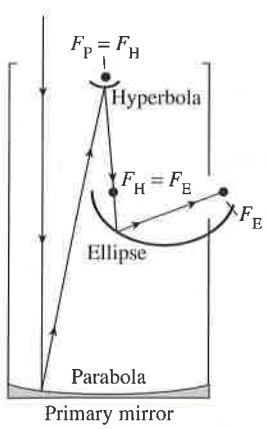


FIGURE 8.29 Cross section of a reflecting telescope.

Reflective Property of a Hyperbola

Like other conics, a hyperbola can be used to make a reflector of sound, light, and other waves. If we rotate a hyperbola in three-dimensional space about its focal axis, the hyperbola sweeps out a **hyperboloid of revolution**. If a signal is directed toward a focus of a reflective hyperboloid, the signal reflects off the hyperbolic surface to the other focus. In Figure 8.29 light reflects off a primary parabolic mirror toward the mirror's focus $F_P = F_H$, which is also the focus of a small hyperbolic mirror. The light is then reflected off the hyperbolic mirror, toward the hyperboloid's other focus $F_H = F_E$, which is also the focus of an elliptical mirror. Finally the light is reflect into the observer's eye, which is at the second focus of the ellipsoid F_E.

Reflecting telescopes date back to the 1600s when Isaac Newton used a primary parabolic mirror in combination with a flat secondary mirror, slanted to reflect the light out the side to the eyepiece. French optician G. Cassegrain was the first to use a hyperbolic secondary mirror, which directed the light through a hole at the vertex of the primary mirror (see Exercise 70). Today, reflecting telescopes such as the Hubble Space Telescope have become quite sophisticated and must have nearly perfect mirrors to focus properly.

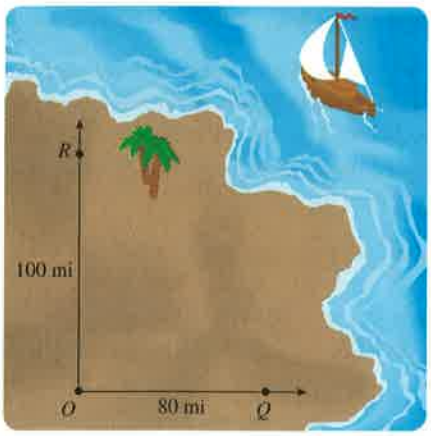


FIGURE 8.30 Strategically located LORAN transmitters O, Q, and R. (Example 6)

Long-Range Navigation

Hyperbolas and radio signals are the basis of the LORAN (long-range navigation) system. Example 6 illustrates this system using the definition of hyperbola and the fact that radio signals travel 980 ft per microsecond (1 microsecond = 1 μsec = 10^{-6} sec).

EXAMPLE 6 Using the LORAN system

Radio signals are sent simultaneously from transmitters located at points O, Q, and R (Figure 8.30). R is 100 mi due north of O, and Q is 80 mi due east of O. The LORAN receiver on sloop *Gloria* receives the signal from O 323.27 μsec after the signal from R, and 258.61 μsec after the signal from Q. What is the sloop's bearing and distance from O?

SOLUTION

Model

The *Gloria* is at a point of intersection between two hyperbolas: one with foci O and R, the other with foci O and Q.

The hyperbola with foci $O(0, 0)$ and $R(0, 100)$ has center $(0, 50)$ and transverse axis

$$2a = (323.27\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec})(1\ \text{mi}/5280\ \text{ft}) \approx 60\ \text{mi}.$$

Thus $a \approx 30$ and $b = \sqrt{c^2 - a^2} \approx \sqrt{50^2 - 30^2} = 40$, yielding the equation

$$\frac{(y-50)^2}{30^2} - \frac{x^2}{40^2} = 1.$$

The hyperbola with foci $O(0, 0)$ and $Q(80, 0)$ has center $(40, 0)$ and transverse axis

$$2a = (258.61\ \mu\text{sec})(980\ \text{ft}/\mu\text{sec})(1\ \text{mi}/5280\ \text{ft}) \approx 48\ \text{mi}.$$

Thus $a \approx 24$ and $b = \sqrt{c^2 - a^2} \approx \sqrt{40^2 - 24^2} = 32$, yielding the equation

$$\frac{(x-40)^2}{24^2} - \frac{y^2}{32^2} = 1.$$

Solve Graphically

The *Gloria* is at point P where upper and right-hand branches of the hyperbolas meet (see Figure 8.31). Using a grapher we find that $P \approx (187.09, 193.49)$. So the bearing from point O is

$$\theta \approx 90° - \tan^{-1}\left(\frac{193.49}{187.09}\right) \approx 44.04°,$$

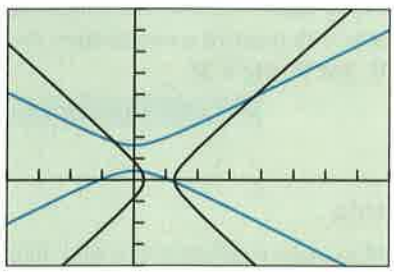
and the distance from point O is

$$d \approx \sqrt{187.09^2 + 193.49^2} \approx 269.15.$$

Interpret

The *Gloria* is about 187.1 mi east and 193.5 mi north of point O on a bearing of roughly 44°. The sloop is about 269 mi from point O.

Now try Exercise 57.



[−200, 400] by [−200, 400]

FIGURE 8.31 Graphs for Example 6.

QUICK REVIEW 8.3

(For help, go to Sections P.2, P.5, and 7.1.)

In Exercises 1 and 2, find the distance between the given points.

1. $(4, -3)$ and $(-7, -8)$

2. $(a, -3)$ and (b, c)

In Exercises 3 and 4, solve for y in terms of x.

3. $\frac{y^2}{16} - \frac{x^2}{9} = 1$ **4.** $\frac{x^2}{36} - \frac{y^2}{4} = 1$

In Exercises 5–8, solve for x algebraically.

5. $\sqrt{3x + 12} - \sqrt{3x - 8} = 10$

6. $\sqrt{4x + 12} - \sqrt{x + 8} = 1$

7. $\sqrt{6x^2 + 12} - \sqrt{6x^2 + 1} = 1$

8. $\sqrt{2x^2 + 12} - \sqrt{3x^2 + 4} = -8$

In Exercises 9 and 10, solve the system of equations.

9. $c - a = 2$ and $c^2 - a^2 = 16a/3$

10. $c - a = 1$ and $c^2 - a^2 = 25a/12$

SECTION 8.3 EXERCISES

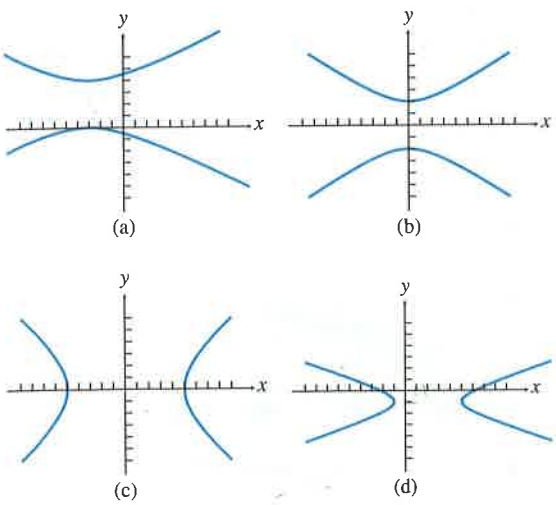
In Exercises 1–6, find the vertices and foci of the hyperbola.

1. $\frac{x^2}{16} - \frac{y^2}{7} = 1$ **2.** $\frac{y^2}{25} - \frac{x^2}{21} = 1$

3. $\frac{y^2}{36} - \frac{x^2}{13} = 1$ **4.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$

5. $3x^2 - 4y^2 = 12$ **6.** $9x^2 - 4y^2 = 36$

In Exercises 7–10, match the graph with its equation.



7. $\frac{x^2}{25} - \frac{y^2}{16} = 1$ **8.** $\frac{y^2}{4} - \frac{x^2}{9} = 1$

9. $\frac{(y-2)^2}{4} - \frac{(x+3)^2}{16} = 1$ **10.** $\frac{(x-2)^2}{9} - (y+1)^2 = 1$

In Exercises 11–16, sketch the graph of the hyperbola by hand.

11. $\frac{x^2}{49} - \frac{y^2}{25} = 1$ **12.** $\frac{y^2}{64} - \frac{x^2}{25} = 1$

13. $\frac{y^2}{25} - \frac{x^2}{16} = 1$ **14.** $\frac{x^2}{169} - \frac{y^2}{144} = 1$

15. $\frac{(x+3)^2}{16} - \frac{(y-1)^2}{4} = 1$ **16.** $\frac{(x-1)^2}{2} - \frac{(y+3)^2}{4} = 1$

In Exercises 17–22, graph the hyperbola using a function grapher.

17. $\frac{x^2}{36} - \frac{y^2}{16} = 1$ **18.** $\frac{y^2}{64} - \frac{x^2}{16} = 1$

19. $\frac{x^2}{4} - \frac{y^2}{9} = 1$ **20.** $\frac{y^2}{16} - \frac{x^2}{9} = 1$

21. $\frac{x^2}{4} - \frac{(y-3)^2}{5} = 1$ **22.** $\frac{(y-3)^2}{9} - \frac{(x+2)^2}{4} = 1$

In Exercises 23–38, find an equation in standard form for the hyperbola that satisfies the given conditions.

23. Foci $(\pm 3, 0)$, transverse axis length 4

24. Foci $(0, \pm 3)$, transverse axis length 4

25. Foci $(0, \pm 15)$, transverse axis length 8

26. Foci $(\pm 5, 0)$, transverse axis length 3

27. Center at $(0, 0)$, $a = 5$, $e = 2$, horizontal focal axis

28. Center at $(0, 0)$, $a = 4$, $e = 3/2$, vertical focal axis

29. Center at $(0, 0)$, $b = 5$, $e = 13/12$, vertical focal axis

30. Center at $(0, 0)$, $c = 6$, $e = 2$, horizontal focal axis

31. Transverse axis endpoints $(2, 3)$ and $(2, -1)$, conjugate axis length 6

32. Transverse axis endpoints $(5, 3)$ and $(-7, 3)$, conjugate axis length 10

33. Transverse axis endpoints $(-1, 3)$ and $(5, 3)$, slope of one asymptote $4/3$

34. Transverse axis endpoints $(-2, -2)$ and $(-2, 7)$, slope of one asymptote $4/3$

35. Foci $(-4, 2)$ and $(2, 2)$, transverse axis endpoints $(-3, 2)$ and $(1, 2)$

36. Foci $(-3, -11)$ and $(-3, 0)$, transverse axis endpoints $(-3, -9)$ and $(-3, -2)$

37. Center at $(-3, 6)$, $a = 5$, $e = 2$, vertical focal axis

38. Center at $(1, -4)$, $c = 6$, $e = 2$, horizontal focal axis

In Exercises 39–42, find the center, vertices, and the foci of the hyperbola.

39. $\dfrac{(x+1)^2}{144} - \dfrac{(y-2)^2}{25} = 1$ **40.** $\dfrac{(x+4)^2}{12} - \dfrac{(y+6)^2}{13} = 1$

41. $\dfrac{(y+3)^2}{64} - \dfrac{(x-2)^2}{81} = 1$ **42.** $\dfrac{(y-1)^2}{25} - \dfrac{(x+5)^2}{11} = 1$

In Exercises 43–46, graph the hyperbola using a parametric grapher in Dot graphing mode.

43. $\dfrac{y^2}{25} - \dfrac{x^2}{4} = 1$ **44.** $\dfrac{x^2}{30} - \dfrac{y^2}{20} = 1$

45. $\dfrac{(x+3)^2}{12} - \dfrac{(y-6)^2}{5} = 1$ **46.** $\dfrac{(y+1)^2}{15} - \dfrac{(x-2)^2}{6} = 1$

In Exercises 47–50, graph the hyperbola, and find its vertices, foci, and eccentricity.

47. $4(y-1)^2 - 9(x-3)^2 = 36$

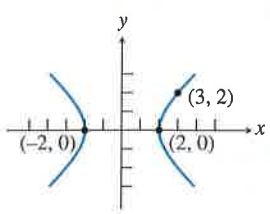
48. $4(x-2)^2 - 9(y+4)^2 = 1$

49. $9x^2 - 4y^2 - 36x + 8y - 4 = 0$

50. $25y^2 - 9x^2 - 50y - 54x - 281 = 0$

In Exercises 51 and 52, write an equation for the hyperbola.

51.



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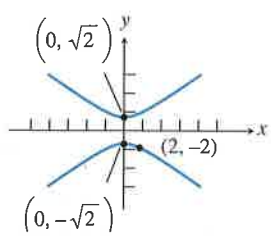


53. Writing to Learn Prove that an equation for the hyperbola with center $(0, 0)$, foci $(0, \pm c)$, and semitransverse axis a is $y^2/a^2 - x^2/b^2 = 1$, where $c > a > 0$ and $b^2 = c^2 - a^2$. [*Hint*: Refer to derivation at the beginning of the section.]

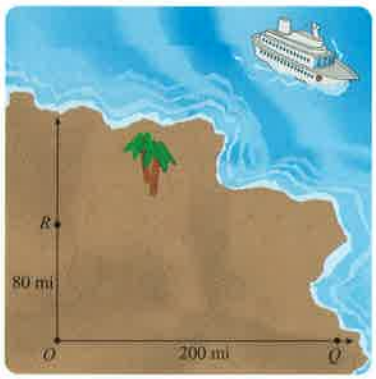
54. Degenerate Hyperbolas Graph the degenerate hyperbola.

(a) $\dfrac{x^2}{4} - y^2 = 0$ **(b)** $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 0$

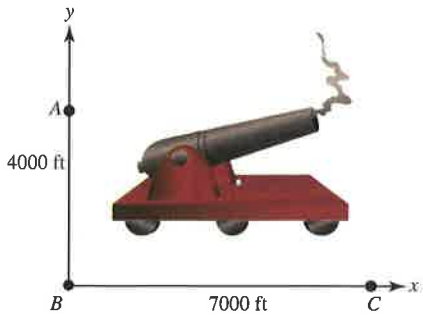
55. Rogue Comet A comet following a hyperbolic path about the Sun has a perihelion of 120 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 250 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is $(0, 0)$ and the Sun lies on the positive x-axis?

56. Rogue Comet A comet following a hyperbolic path about the Sun has a perihelion of 140 Gm. When the line from the comet to the Sun is perpendicular to the focal axis of the orbit, the comet is 405 Gm from the Sun. Calculate a, b, c, and e. What are the coordinates of the center of the Sun if the center of the hyperbolic orbit is $(0, 0)$ and the Sun lies on the positive x-axis?

57. Long-Range Navigation Three LORAN radio transmitters are positioned as shown in the figure, with R due north of O and Q due east of O. The cruise ship *Princess Ann* receives simultaneous signals from the three transmitters. The signal from O arrives 323.27 μsec after the signal from R, and 646.53 μsec after the signal from Q. Determine the ship's bearing and distance from point O.



58. Gun Location Observers are located at positions A, B, and C with A due north of B. A cannon is located somewhere in the first quadrant as illustrated in the figure. A hears the sound of the cannon 2 sec before B, and C hears the sound 4 sec before B. Determine the bearing and distance of the cannon from point B. (Assume that sound travels at 1100 ft/sec.)



Group Activities In Exercises 59 and 60, solve the system of equations algebraically and support your answer graphically.

59. $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

$x - \dfrac{2\sqrt{3}}{3}y = -2$

60. $\dfrac{x^2}{4} - y^2 = 1$

$x^2 + y^2 = 9$

61. Group Activity Consider the system of equations

$$\frac{x^2}{4} - \frac{y^2}{25} = 1$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

(a) Solve the system graphically.

(b) If you have access to a grapher that also does symbolic algebra, use it to find the the exact solutions to the system.

62. Writing to Learn Escape of the Unbound When NASA launches a space probe, the probe reaches a speed sufficient for it to become unbound from Earth and escape along a hyperbolic trajectory. Look up *escape speed* in an astronomy textbook or on the Internet, and write a paragraph in your own words about what you find.

Standardized Test Questions

63. True or False The distance from a focus of a hyperbola to the closer vertex is $a(e - 1)$, where a is the semitransverse axis and e is the eccentricity. Justify your answer.

64. True or False Unlike that of an ellipse, the Pythagorean relation for a hyperbola is the usual $a^2 + b^2 = c^2$. Justify your answer.

In Exercises 65–68, you may use a graphing calculator to solve the problem.

65. Multiple Choice One focus of $x^2 - 4y^2 = 4$ is

(a) $(4, 0)$.

(b) $(\sqrt{5}, 0)$.

(c) $(2, 0)$.

(d) $(\sqrt{3}\ 0)$.

(e) $(1, 0)$.

66. Multiple Choice The focal axis of $\frac{(x+5)^2}{9} - \frac{(y-6)^2}{16} = 1$ is

(a) $y = 2$.

(b) $y = 3$.

(c) $y = 4$.

(d) $y = 5$.

(e) $y = 6$.

67. Multiple Choice The center of $4x^2 - 12y^2 - 16x - 72y - 44 = 0$ is

(a) $(2, -2)$.

(b) $(2, -3)$.

(c) $(2, -4)$.

(d) $(2, -5)$.

(e) $(2, -6)$.

68. Multiple Choice The slopes of the asymptotes of the hyperbola $\frac{x^2}{4} - \frac{y^2}{3} = 1$ are

(a) ± 1.

(b) $\pm 3/2$.

(c) $\pm\sqrt{3}/2$.

(d) $\pm 2/3$.

(e) $\pm 4/3$.

Explorations

69. Constructing Points of a Hyperbola Use a geometry software package, such as *Cabri Geometry II*™, *The Geometer's Sketchpad*®, or a similar application on a hand-held device, to carry out the following construction.

(a) Start by placing the coordinate axes in the construction window.

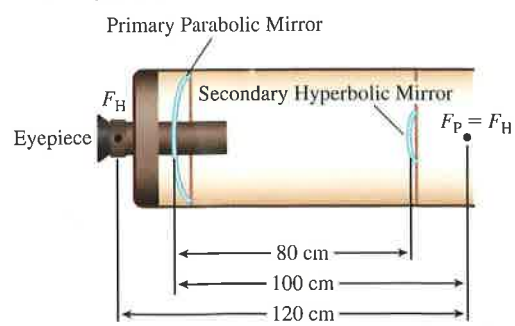
(b) Construct two points on the x-axis at $(\pm 5, 0)$ as the foci.

(c) Construct concentric circles of radii $r = 1, 2, 3, \ldots, 12$ centered at these two foci.

(d) Construct the points where these concentric circles meet and have a difference of radii of $2a = 6$, and overlay the conic that passes through these points if the software has a conic tool.

(e) Find the equation whose graph includes all of these points.

70. Cassegrain Telescope A Cassegrain telescope as described in the section has the dimensions shown in the figure. Find the standard form for the equation of the hyperbola centered at the origin with the focal axis the x-axis.



Extending the Ideas

71. Prove that a nondegenerate graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is a hyperbola if $AC < 0$.

72. Writing to Learn The graph of the equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0$$

is considered to be a degenerate hyperbola. Describe the graph. How is it like a full-fledged hyperbola, and how is it different?

73. Conjugate Hyperbolas The hyperbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \text{ and } \frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$$

obtained by switching the order of subtraction in their standard equations are **conjugate hyperbolas**. Prove that these hyperbolas have the same asymptotes and that the conjugate axis of each of these hyperbolas is the transverse axis of the other hyperbola.

74. Focal Width of a Hyperbola Prove that, for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

if $x = c$, then $y = \pm b^2/a$. Why is it reasonable to define the **focal width** of such hyperbolas to be $2b^2/a$?

75. Writing to Learn Explain how the standard form equations for the conics are related to

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

8.4 TRANSLATION AND ROTATION OF AXES

What you'll learn about

- Second-Degree Equations in Two Variables
- Translating Axes Versus Translating Graphs
- Rotation of Axes
- Discriminant Test

. . . and why

You will see ellipses, hyperbolas, and parabolas as members of the family of conic sections rather than as separate types of curves.

Second-Degree Equations in Two Variables

In Section 8.1, we began with a unified approach to conic sections, learning that parabolas, ellipses, and hyperbolas are all cross sections of a right circular cone. In Sections 8.1–8.3, we gave separate plane-geometry definitions for parabolas, ellipses, and hyperbolas that led to separate kinds of equations for each type of curve. In this section and the next, we once again consider parabolas, ellipses, and hyperbolas as a unified family of interrelated curves.

In Section 8.1, we claimed that the conic sections can be defined algebraically in the Cartesian plane as the graphs of *second-degree equations in two variables*, that is, equations of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, and C are not all zero. In this section, we investigate equations of this type, which are really just *quadratic equations in x and y*. Because they are quadratic equations, we can adapt familiar methods to this unfamiliar setting. That is exactly what we do in Examples 1–3.

EXAMPLE 1 Graphing a second-degree equation

Solve for y, and use a function grapher to graph

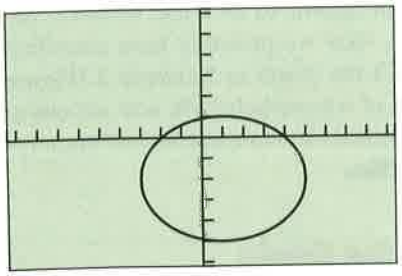
$$9x^2 + 16y^2 - 18x + 64y - 71 = 0.$$

SOLUTION Rearranging terms yields the equation:

$$16y^2 + 64y + (9x^2 - 18x - 71) = 0.$$

The quadratic formula gives us

$$\begin{aligned} y &= \frac{-64 \pm \sqrt{64^2 - 4(16)(9x^2 - 18x - 71)}}{2(16)} \\ &= \frac{-8 \pm 3\sqrt{-x^2 + 2x + 15}}{4} \\ &= -2 \pm \frac{3}{4}\sqrt{-x^2 + 2x + 15} \end{aligned}$$



[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 8.32 The graph of $9x^2 + 16y^2 - 18x + 64y - 71 = 0$. (Example 1)

Let

$$Y1 = -2 + 0.75\sqrt{-x^2 + 2x + 15} \text{ and } Y2 = -2 - 0.75\sqrt{-x^2 + 2x + 15},$$

and graph the two equations in the same viewing window, as shown in Figure 8.32. The combined figure appears to be an ellipse.

Now try Exercise 1.

In the equation in Example 1, there was no Bxy term. None of the examples in Sections 8.1–8.3 included such a *cross-product* term. A cross-product term in the equation causes the graph to tilt relative to the coordinate axes, as illustrated in Examples 2 and 3.

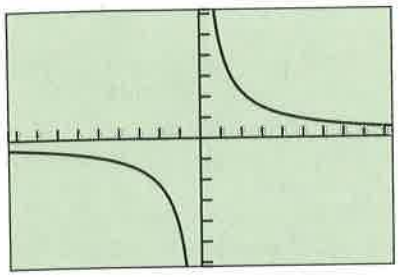
EXAMPLE 2 Graphing a second-degree equation

Solve for y, and use a function grapher to graph

$$2xy - 9 = 0.$$

SOLUTION This equation can be rewritten as $2xy = 9$ or as $y = 9/(2x)$. The graph of this equation is shown in Figure 8.33. It appears to be a hyperbola with a slant focal axis.

Now try Exercise 5.



[–9.4, 9.4] by [–6.2, 6.2]

FIGURE 8.33 The graph of $2xy - 9 = 0$. (Example 2)

EXAMPLE 3 Graphing a second-degree equation

Solve for y, and use a function grapher to graph

$$x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0.$$

SOLUTION We rearrange the terms as a quadratic equation in y:

$$4y^2 + (4x - 90)y + (x^2 - 30x + 450) = 0.$$

The quadratic formula gives us

$$y = \frac{-(4x - 90) \pm \sqrt{(4x - 90)^2 - 4(4)(x^2 - 30x + 450)}}{2(4)}$$

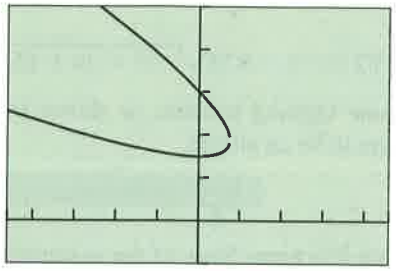
$$= \frac{45 - 2x \pm \sqrt{225 - 60x}}{4}$$

Let

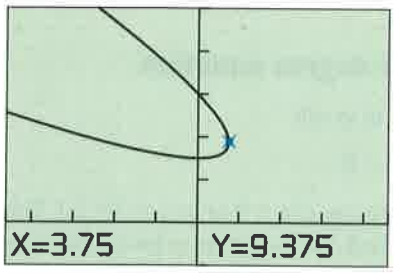
$$Y1 = \frac{45 - 2x + \sqrt{225 - 60x}}{4} \text{ and } Y2 = \frac{45 - 2x - \sqrt{225 - 60x}}{4},$$

and graph the two equations in the same viewing window, as shown in Figure 8.34a. The combined figure appears to be a parabola, with a slight gap due to grapher failure. The combined graph should connect at a point for which the radicand $225 - 60x = 0$, that is, when $x = 225/60 = 15/4 = 3.75$. Figure 8.34b supports this analysis.

Now try Exercise 9.



[–23, 23] by [–5, 25]
(a)



[–23, 23] by [–5, 25]
(b)

FIGURE 8.34 The graph of $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ (a) with a gap and (b) with the trace feature activated at the connecting point. (Example 3)

The graphs obtained in Examples 1–3 all *appear* to be conic sections, but how can we be sure? If they are conics, then we probably have classified Examples 1 and 2 correctly, but couldn't the graph in Example 3 (Figure 8.34) be part of an ellipse or one branch of a hyperbola? We now set out to answer these questions and to develop methods for simplifying and classifying second-degree equations in two variables.

Translating Axes versus Translating Graphs

The coordinate axes are often viewed as a permanent fixture of the plane, but this just isn't so. We can shift the position of axes just as we have been shifting the position of graphs since Chapter 1. Such a **translation of axes** produces a new set of axes parallel to the original axes, as shown in Figure 8.35.

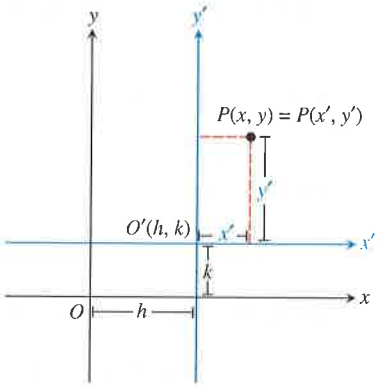


FIGURE 8.35 A translation of Cartesian coordinate axes.

Figure 8.35 shows a plane containing a point P that is named in two ways: using the coordinates (x, y) and the coordinates (x', y'). The coordinates (x, y) are based on the original x- and y- axes and the original origin O, while (x', y') are based on the translated x'- and y'-axes and the corresponding origin O'.

Translation-of-Axes Formulas

The coordinates (x, y) and (x', y') based on parallel sets of axes are related by either of the following **translation formulas**:

$$x = x' + h \quad \text{and} \quad y = y' + k$$

or

$$x' = x - h \quad \text{and} \quad y' = y - k.$$

We use the second pair of translation formulas in Example 4.

EXAMPLE 4 Revisiting Example 1

Prove that $9x^2 + 16y^2 - 18x + 64y - 71 = 0$ is the equation of an ellipse. Translate the coordinate axes so that the origin is at the center of this ellipse.

SOLUTION We complete the square of both x and y:

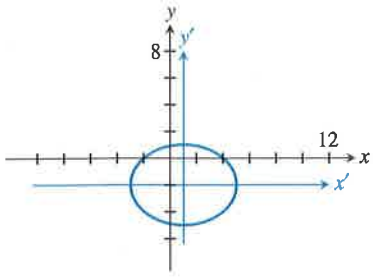


FIGURE 8.36 The graph of $(x')^2/16 + (y')^2/9 = 1$. (Example 4)

$$9x^2 - 18x + 16y^2 + 64y = 71$$
$$9(x^2 - 2x + 1) + 16(y^2 + 4y + 4) = 71 + 9(1) + 16(4)$$
$$9(x - 1)^2 + 16(y + 2)^2 = 144$$
$$\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$$

This is a standard equation of an ellipse. If we let $x' = x - 1$ and $y' = y + 2$, then the equation of the ellipse becomes

$$\frac{(x')^2}{16} + \frac{(y')^2}{9} = 1.$$

Figure 8.36 shows the graph of this final equation in the new $x'y'$ coordinate system, with the original xy-axes overlaid. Compare Figures 8.32 and 8.36.

Now try Exercise 21.

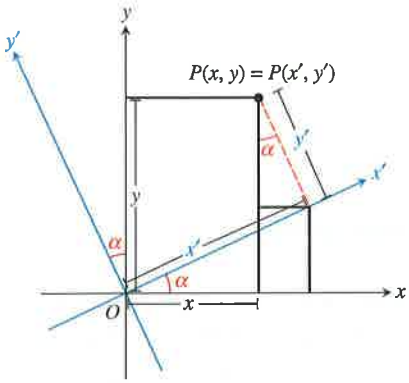


FIGURE 8.37 A rotation of Cartesian coordinate axes.

Rotation of Axes

To show that the equation in Example 2 or 3 is the equation of a conic section, we need to rotate the coordinate axes so that one axis aligns with the (focal) axis of the conic. In such a **rotation of axes**, the origin stays fixed, and we rotate the x- and y-axes through an angle α to obtain the x'- and y'-axes. (See Figure 8.37.)

Figure 8.37 shows a plane containing a point P named in two ways: as (x, y) and as (x', y'). The coordinates (x, y) are based on the original x- and y-axes, while (x', y') are based on the rotated x'- and y'-axes.

Rotation-of-Axes Formulas

The coordinates (x, y) and (x', y') based on rotated sets of axes are related by either of the following **rotation formulas**:

$$x' = x\cos\alpha + y\sin\alpha \quad \text{and} \quad y' = -x\sin\alpha + y\cos\alpha,$$

or

$$x = x'\cos\alpha - y'\sin\alpha \quad \text{and} \quad y = x'\sin\alpha + y'\cos\alpha.$$

where α, $0 < \alpha < \pi/2$, is the *angle of rotation*.

The first pair of equations was established in Example 10 of Section 7.2. The second pair can be derived directly from the geometry of Figure 8.37 (see Exercise 55) and is used in Example 5.

EXAMPLE 5 Revisiting Example 2

Prove that $2xy - 9 = 0$ is the equation of a hyperbola by rotating the coordinate axes through an angle $\alpha = \pi/4$.

SOLUTION Because $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, the rotation equations become

$$x = \frac{x' - y'}{\sqrt{2}} \quad \text{and} \quad y = \frac{x' + y'}{\sqrt{2}}.$$

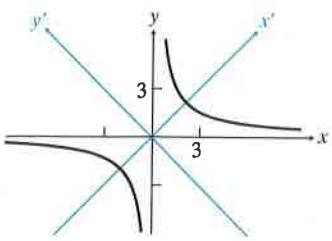


FIGURE 8.38 The graph of $2xy - 9 = 0$. (Example 5)

So by rotating the axes, the equation $2xy - 9 = 0$ becomes

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 9 = 0$$

$$(x')^2 - (y')^2 - 9 = 0$$

To see that this is the equation of a hyperbola, we put it in standard form:

$$(x')^2 - (y')^2 = 9$$

$$\frac{(x')^2}{9} - \frac{(y')^2}{9} = 1$$

Figure 8.38 shows the graph of the original equation in the original xy system with the $x'y'$-axes overlaid. Now try Exercise 35.

In Example 5 we converted a second-degree equation in x and y into a second-degree equation in x' and y' using the rotation formulas. By choosing the angle of rotation appropriately, there was no $x'y'$ cross-product term in the final equation, which allowed us to put it in standard form. We now generalize this process.

Coefficients for a Conic in a Rotated System

If we apply the rotation formulas to the general second-degree equation in x and y, we obtain a second-degree equation in x' and y' of the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0,$$

where the coefficients are

$$A' = A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha$$

$$B' = B\cos 2\alpha + (C - A)\sin 2\alpha$$

$$C' = C\cos^2\alpha - B\cos\alpha\sin\alpha + A\sin^2\alpha$$

$$D' = D\cos\alpha + E\sin\alpha$$

$$E' = E\cos\alpha - D\sin\alpha$$

$$F' = F$$

In order to eliminate the cross-product term and thus align the coordinate axes with the focal axis of the conic, we rotate the coordinate axes through an angle α that causes B' to equal 0. Setting $B' = B\cos 2\alpha + (C - A)\sin 2\alpha = 0$ leads to the following useful result.

Angle of Rotation to Eliminate the Cross-Product Term

If $B \neq 0$, an angle of rotation α such that

$$\cot 2\alpha = \frac{A - C}{B} \text{ and } 0 < \alpha < \frac{\pi}{2}$$

will eliminate the term $B'x'y'$ from the second-degree equation in the rotated $x'y'$ coordinate system.

EXAMPLE 6 Revisiting Example 3

Prove that $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ is the equation of a parabola by rotating the coordinate axes through a suitable angle α.

SOLUTION The angle of rotation α must satisfy the equation

$$\cot 2\alpha = \frac{A - C}{B} = \frac{1 - 4}{4} = -\frac{3}{4}.$$

So

$$\cos 2\alpha = -\frac{3}{5},$$

and thus

$$\cos \alpha = \sqrt{\frac{1 + \cos 2\alpha}{2}} = \sqrt{\frac{1 + (-3/5)}{2}} = \frac{1}{\sqrt{5}},$$

$$\sin \alpha = \sqrt{\frac{1 - \cos 2\alpha}{2}} = \sqrt{\frac{1 - (-3/5)}{2}} = \frac{2}{\sqrt{5}}.$$

Therefore the coefficients of the transformed equation are

$$A' = 1 \cdot \frac{1}{5} + 4 \cdot \frac{2}{5} + 4 \cdot \frac{4}{5} = \frac{25}{5} = 5$$

$$B' = 0$$

$$C' = 4 \cdot \frac{1}{5} - 4 \cdot \frac{2}{5} + 1 \cdot \frac{4}{5} = 0$$

$$D' = -30 \cdot \frac{1}{\sqrt{5}} - 90 \cdot \frac{2}{\sqrt{5}} = -\frac{210}{\sqrt{5}} = -42\sqrt{5}$$

$$E' = -90 \cdot \frac{1}{\sqrt{5}} + 30 \cdot \frac{2}{\sqrt{5}} = -\frac{30}{\sqrt{5}} = -6\sqrt{5}$$

$$F' = 450$$

So the equation $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$ becomes

$$5x'^2 - 42\sqrt{5}x' - 6\sqrt{5}y' + 450 = 0.$$

After completing the square of the x-terms, the equation becomes

$$\left(x' - \frac{21}{\sqrt{5}}\right)^2 = \frac{6}{\sqrt{5}}\left(y' - \frac{3\sqrt{5}}{10}\right).$$

If we translate using $h = 21/\sqrt{5}$ and $k = 3\sqrt{5}/10$, then the equation becomes

$$(x'')^2 = \frac{6}{\sqrt{5}}(y''),$$

a standard equation of a parabola.

Figure 8.39 shows the graph of the original equation in the original xy coordinate system, with the $x''y''$-axes overlaid. Now try Exercise 39.

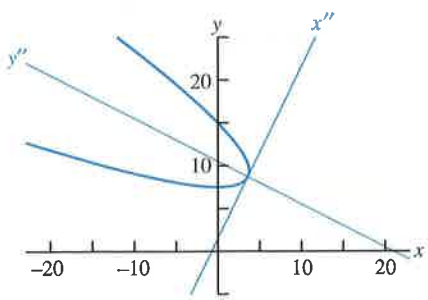


FIGURE 8.39 The graph of $x^2 + 4xy + 4y^2 - 30x - 90y + 450 = 0$. (Example 6)

Discriminant Test

Example 6 demonstrates that the algebra of rotation can get ugly. Fortunately, we can determine which type of conic a second-degree equation represents by looking at the sign of the **discriminant** $B^2 - 4AC$.

Discriminant Test

The second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ graphs as

- a hyperbola if $B^2 - 4AC > 0$,
- a parabola if $B^2 - 4AC = 0$,
- an ellipse if $B^2 - 4AC < 0$,

except for degenerate cases.

This test hinges on the fact that the discriminant $B^2 - 4AC$ is **invariant under rotation**; in other words, even though A, B, and C do change when we rotate the coordinate axes, the combination $B^2 - 4AC$ maintains its value.

EXAMPLE 7 Revisiting Examples 5 and 6

(a) In Example 5, before the rotation $B^2 - 4AC = (2)^2 - 4(0)(0) = 4$, and after the rotation $B'^2 - 4A'C' = (0)^2 - 4(1)(-1) = 4$. The positive discriminant tells us the conic is a hyperbola.

(b) In Example 6, before the rotation $B^2 - 4AC = (4)^2 - 4(1)(4) = 0$, and after the rotation $B'^2 - 4A'C' = (0)^2 - 4(5)(0) = 0$. The zero discriminant tells us the conic is a parabola. Now try Exercise 43.

Not only is the discriminant $B^2 - 4AC$ invariant under rotation, but also its *sign* is invariant under translation and under algebraic manipulations that preserve the equivalence of the equation, such as multiplying both sides of the equation by a nonzero constant.

The discriminant test can be applied to degenerate conics. Table 8.2 displays the three basic types of conic sections grouped with their associated degenerate conics. Each conic or degenerate conic is shown with a sample equation and the sign of its discriminant.

Conics and the Equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Conic	Sample Equation	A	B	C	D	E	F	Sign of Discriminant
Hyperbola	$x^2 - 2y^2 = 1$	1		-2			-1	Positive
Intersecting lines	$x^2 + xy = 0$	1	1					Positive
Parabola	$x^2 = 2y$	1				-2		Zero
Parallel lines	$x^2 = 4$	1					-4	Zero
One line	$y^2 = 0$			1				Zero
No graph	$x^2 = -1$	1					1	Zero
Ellipse	$x^2 + 2y^2 = 1$	1		2			-1	Negative
Circle	$x^2 + y^2 = 9$	1		1			-9	Negative
Point	$x^2 + y^2 = 0$	1		1				Negative
No graph	$x^2 + y^2 = -1$	1		1			1	Negative

QUICK REVIEW 8.4

(For help, go to Sections 4.7 and 5.4.)

In Exercises 1–10, assume $0 \le \alpha < \pi/2$.

1. Given that $\cot 2\alpha = 5/12$, find $\cos 2\alpha$.
2. Given that $\cot 2\alpha = 8/15$, find $\cos 2\alpha$.
3. Given that $\cot 2\alpha = 1/\sqrt{3}$, find $\cos 2\alpha$.
4. Given that $\cot 2\alpha = 2/\sqrt{5}$, find $\cos 2\alpha$.
5. Given that $\cot 2\alpha = 0$, find α.
6. Given that $\cot 2\alpha = \sqrt{3}$, find α.
7. Given that $\cot 2\alpha = 3/4$, find $\cos \alpha$.
8. Given that $\cot 2\alpha = 3/\sqrt{7}$, find $\cos \alpha$.
9. Given that $\cot 2\alpha = \sqrt{11}/5$, find $\sin \alpha$.
10. Given that $\cot 2\alpha = 45/28$, find $\sin \alpha$.

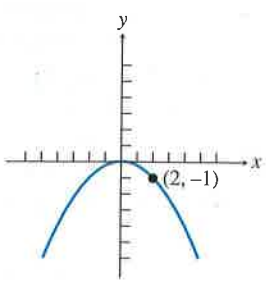
SECTION 8.4 EXERCISES

In Exercises 1–12, solve for y, and use a function grapher to graph the conic.

1. $x^2 + y^2 - 6x + 10y + 18 = 0$
2. $4x^2 + y^2 + 24x - 2y + 21 = 0$
3. $y^2 - 8x - 8y + 8 = 0$
4. $x^2 - 4y^2 + 6x - 40y + 91 = 0$
5. $-4xy + 16 = 0$
6. $2xy + 6 = 0$
7. $xy - y - 8 = 0$
8. $2x^2 - 5xy + y = 0$
9. $2x^2 - xy + 3y^2 - 3x + 4y - 6 = 0$
10. $-x^2 + 3xy + 4y^2 - 5x - 10y - 20 = 0$
11. $2x^2 - 4xy + 8y^2 - 10x + 4y - 13 = 0$
12. $2x^2 - 4xy + 2y^2 - 5x + 6y - 15 = 0$

In Exercises 13–16, write an equation in standard form for the conic shown.

13.



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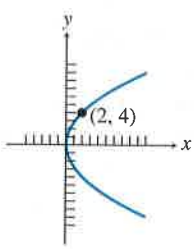

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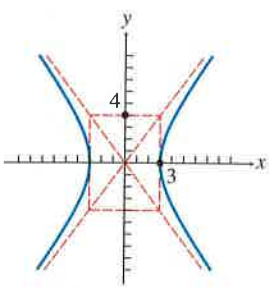

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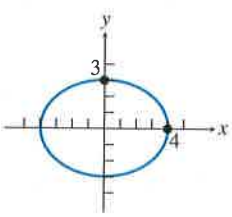


In Exercises 17–20, using the point $P(x, y)$ and the translation information, find the coordinates of P in the translated $x'y'$ coordinate system.

17. $P(x, y) = (2, 3)$, $h = -2$, $k = 4$

18. $P(x, y) = (-2, 5)$, $h = -4$, $k = -7$

19. $P(x, y) = (6, -3)$, $h = 1$, $k = \sqrt{5}$

20. $P(x, y) = (-5, -4)$, $h = \sqrt{2}$, $k = -3$

In Exercises 21–30, identify the type of conic, write the equation in standard form, translate the conic to the origin, and sketch it in the translated coordinate system.

21. $4y^2 - 9x^2 - 18x - 8y - 41 = 0$

22. $2x^2 + 3y^2 + 12x - 24y + 60 = 0$

23. $x^2 + 2x - y + 3 = 0$

24. $3x^2 - 6x - 6y + 10 = 0$

25. $9x^2 + 4y^2 - 18x + 16y - 11 = 0$

26. $16x^2 - y^2 - 32x - 6y - 57 = 0$

27. $y^2 - 4y - 8x + 20 = 0$

28. $2x^2 - 4x + y^2 - 6y = 9$

29. $2x^2 - y^2 + 4x + 6 = 0$

30. $y^2 - 2y + 4x - 12 = 0$

31. **Writing to Learn** **Translation Formulas** Use the geometric relationships illustrated in Figure 8.35 to explain the translation formulas $x = x' + h$ and $y = y' + k$.

32. **Translation Formulas** Prove that if $x = x' + h$ and $y = y' + k$, then $x' = x - h$ and $y' = y - k$.

In Exercises 33–36, using the point $P(x, y)$ and the rotation information, find the coordinates of P in the rotated $x'y'$ coordinate system.

33. $P(x, y) = (-2, 5)$, $\alpha = \pi/4$

34. $P(x, y) = (6, -3)$, $\alpha = \pi/3$

35. $P(x, y) = (-5, -4)$, $\cot 2\alpha = -3/5$

36. $P(x, y) = (2, 3)$, $\cot 2\alpha = 0$

In Exercises 37–40, identify the type of conic, and rotate the coordinate axes to eliminate the xy term. Write and graph the transformed equation.

37. $xy = 8$

38. $3xy + 15 = 0$

39. $2x^2 + \sqrt{3}xy + y^2 - 10 = 0$

40. $3x^2 + 2\sqrt{3}xy + y^2 - 14 = 0$

In Exercises 41 and 42, identify the type of conic, solve for y, and graph the conic. Approximate the angle of rotation needed to eliminate the xy term.

41. $16x^2 - 20xy + 9y^2 - 40 = 0$

42. $4x^2 - 6xy + 2y^2 - 3x + 10y - 6 = 0$

In Exercises 43–52, use the discriminant $B^2 - 4AC$ to decide whether the equation represents a parabola, an ellipse, or a hyperbola.

43. $x^2 - 4xy + 10y^2 + 2y - 5 = 0$

44. $x^2 - 4xy + 3x + 25y - 6 = 0$

45. $2x^2 - xy + 9y^2 - 7x - 3y = 0$

46. $-xy + 3y^2 - 4x + 2y + 8 = 0$

47. $8x^2 - 4xy + 2y^2 + 6 = 0$

48. $3x^2 - 12xy + 4y^2 + x - 5y - 4 = 0$

49. $x^2 - 3y^2 - y - 22 = 0$

50. $5x^2 + 4xy + 3y^2 + 2x + y = 0$

51. $4x^2 - 2xy + y^2 - 5x + 18 = 0$

52. $6x^2 - 4xy + 9y^2 - 40x + 20y - 56 = 0$

53. **Revisiting Example 5** Using the results of Example 5, find the center, vertices, and foci of the hyperbola $2xy - 9 = 0$ in the original coordinate system.

54. **Revisiting Examples 3 and 6** Use information from Examples 3 and 6

(a) to prove that the point $P(x, y) = (3.75, 9.375)$ where the graphs of $Y1 = (45 - 2x + \sqrt{225 - 60x})/4$ and $Y2 = (45 - 2x - \sqrt{225 - 60x})/4$ meet is not the vertex of the parabola,

(b) to prove that the point $V(x, y) = (3.6, 8.7)$ is the vertex of the parabola.

55. Rotation Formulas Prove $x = x' \cos \alpha - y' \sin \alpha$ and $y = x' \sin \alpha + y' \cos \alpha$ using the geometric relationships illustrated in Figure 8.37.

56. Rotation Formulas Prove that if $x' = x \cos \alpha + y \sin \alpha$ and $y' = -x \sin \alpha + y \cos \alpha$, then $x = x' \cos \alpha - y' \sin \alpha$ and $y = x' \sin \alpha + y' \cos \alpha$.

Standardized Test Questions

57. True or False The graph of the equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ (A and C not both zero) has a focal axis aligned with the coordinate axes. Justify your answer.

58. True or False The graph of the equation $x^2 + y^2 + Dx + Ey + F = 0$ is a circle or a degenerate circle. Justify your answer.

In Exercises 59–62, solve the problem without using a calculator.

59. Multiple Choice Which of the following is **not** a reason to translate the axes of a conic?

(a) to simplify its equation

(b) to eliminate the cross-product term

(c) to place its center or vertex at the origin

(d) to make it easier to identify its type

(e) to make it easier to sketch by hand

60. Multiple Choice Which of the following is **not** a reason to rotate the axes of a conic?

(a) to simplify its equation

(b) to eliminate the cross-product term

(c) to place its center or vertex at the origin

(d) to make it easier to identify its type

(e) to make it easier to sketch by hand

61. Multiple Choice The vertices of $9x^2 + 16y^2 - 18x + 64y - 71 = 0$ are

(a) $(1 \pm 4, -2)$

(b) $(1 \pm 3, -2)$

(c) $(4 \pm 1, 3)$

(d) $(4 \pm 2, 3)$

(e) $(1, -2 \pm 3)$

62. Multiple Choice The asymptotes of the hyperbola $xy = 4$ are

(a) $y = x, y = -x$.

(b) $y = 2x, y = -\frac{x}{2}$.

(c) $y = -2x, y = \frac{x}{2}$.

(d) $y = 4x, y = -\frac{x}{4}$.

(e) the coordinate axes

Explorations

63. Axes of Oblique Conics The axes of conics that are not aligned with the coordinate axes are often included in the graphs of conics.

(a) Recreate the graph shown in Figure 8.38 using a function grapher including the x'- and y'-axes. What are the equations of these rotated axes?

(b) Recreate the graph shown in Figure 8.39 using a function grapher including the x''- and y''-axes. What are the equations of these rotated and translated axes?

64. The Discriminant Determine what happens to the sign of $B^2 - 4AC$ within the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

when

(a) the axes are translated h units horizontally and k units vertically,

(b) both sides of the equation are multiplied by the same nonzero constant k.

Extending the Ideas

65. Group Activity Working together prove that the formulas for the coefficients A', B', C', D', E', and F' in a rotated system given on page 670 are correct.

66. Identifying a Conic Develop a way to decide whether $Ax^2 + Cy^2 + Dx + Ey + F = 0$, with A and C not both 0, represents a parabola, an ellipse, or a hyperbola. Write an example to illustrate each of the three cases.

67. Rotational Invariant Prove that $B'^2 - 4A'C' = B^2 - 4AC$ when the xy coordinate system is rotated through an angle α.

68. Other Rotational Invariants Prove each of the following are invariants under rotation:

(a) F, **(b)** $A + C$, **(c)** $D^2 + E^2$.

69. Degenerate Conics Graph all of the degenerate conics listed on page 673. Recall that *degenerate cones* occur when the generator and axis of the cone are parallel or perpendicular. (See Figure 8.2.) Explain the occurrence of all of the degenerate conics listed on the basis of cross sections of typical or degenerate right circular cones.

8.5 POLAR EQUATIONS OF CONICS

What you'll learn about

- Eccentricity Revisited
- Writing Polar Equations for Conics
- Analyzing Polar Equations of Conics
- Orbits Revisited

. . . and why

You will learn the approach to conics used by astronomers.

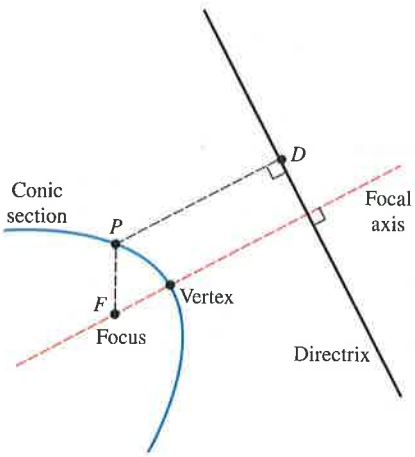


FIGURE 8.40 The geometric structure of a conic section.

Eccentricity Revisited

Eccentricity and polar coordinates provide ways to see once again that parabolas, ellipses, and hyperbolas are a unified family of interrelated curves. We can define these three curves simultaneously by generalizing the focus-directrix definition of parabola given in Section 8.1.

Focus-Directrix Definition Conic Section

A **conic section** is the set of all points in a plane whose distances from a particular point (the **focus**) and a particular line (the **directrix**) in the plane have a constant ratio. (We assume that the focus does not lie on the directrix.)

The line passing through the focus and perpendicular to the directrix is the **(focal) axis** of the conic section. The axis is a line of symmetry for the conic. The point where the conic intersects its axis is a **vertex** of the conic. If P is a point of the conic, F is the focus, and D is the point of the directrix closest to P, then the constant ratio PF/PD is the **eccentricity** e of the conic (see Figure 8.40). A parabola has one focus and one directrix. Ellipses and hyperbolas have two focus-directrix pairs, and either focus-directrix pair can be used with the eccentricity to generate the entire conic section.

Focus-Directrix-Eccentricity Relationship

If P is a point of a conic section, F is the conic's focus, and D is the point of the directrix closest to P, then

$$e = \frac{PF}{PD} \quad \text{and} \quad PF = e \cdot PD,$$

where the constant e is the eccentricity of the conic. Moreover, the conic is

- a hyperbola if $e > 1$,
- a parabola if $e = 1$,
- an ellipse if $e < 1$.

In this approach to conic sections, the eccentricity e is a strictly positive constant, and there are no circles or other degenerate conics.

REMARKS

- To be consistent with our work on parabolas, we could use $2p$ for the distance from the focus to the directrix, but following George B. Thomas, Jr. we use k for this distance. This simplifies our polar equations of conics.
- Rather than religiously using polar coordinates and equations, we use a mixture of the polar and Cartesian systems. So, for example, we use $x = k$ for the directrix rather than $r \cos \theta = k$ or $r = k \sec \theta$.

Writing Polar Equations for Conics

Our focus-directrix definition of conics works best in combination with polar coordinates. Recall that in polar coordinates the origin is the *pole* and the x-axis is the *polar axis*. To obtain a polar equation for a conic section, we position the pole at the conic's focus and the polar axis along the focal axis with the directrix to the right of the pole (Figure 8.41). If the distance from the focus to the directrix is k, the Cartesian equation of the directrix is $x = k$. From Figure 8.41, we see that

$$PF = r \quad \text{and} \quad PD = k - r \cos \theta.$$

So the equation $PF = e \cdot PD$ becomes

$$r = e(k - r \cos \theta),$$

which when solved for r is

$$r = \frac{ke}{1 + e \cos \theta}.$$

In Exercise 53, you are asked to show that this equation is still valid if $r < 0$ or $r \cos \theta > k$. This one equation can produce all sizes and shapes of nondegenerate conic sections. Figure 8.42 shows three typical graphs for this equation. In Exploration 1, you will investigate how changing the value of e affects the graph of $r = ke/(1 + e \cos \theta)$.

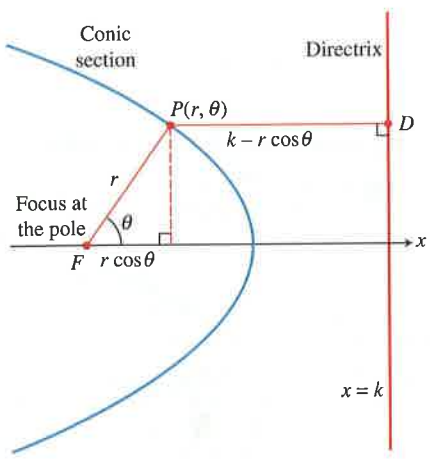


FIGURE 8.41 A conic section in the polar plane.

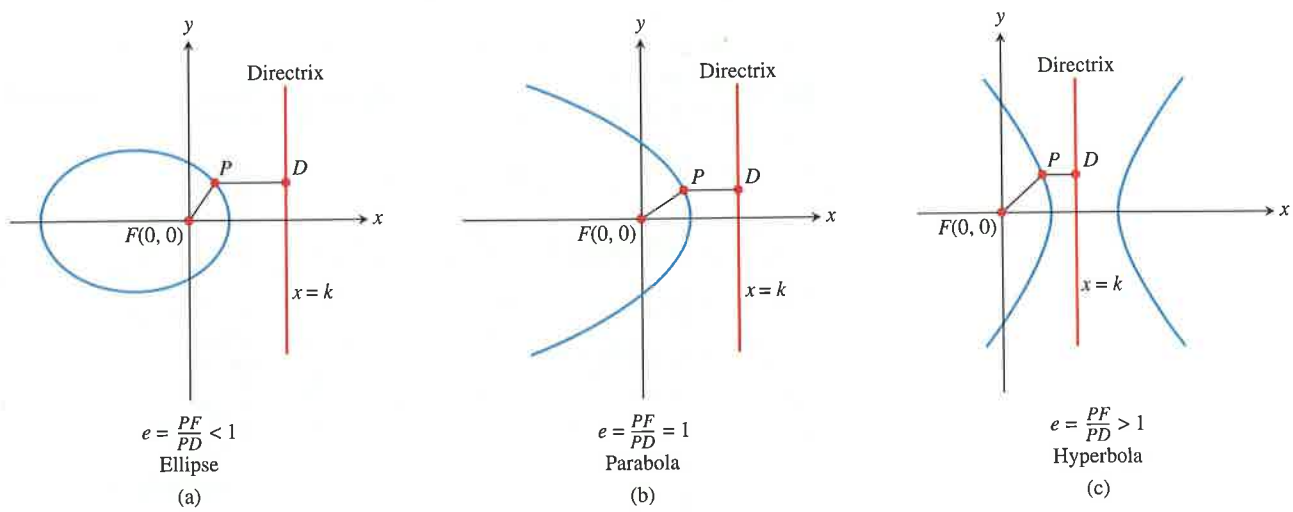


FIGURE 8.42 The three types of conics possible for $r = ke/(1 + e \cos \theta)$.

EXPLORATION 1 Graphing Polar Equations of Conics

Set your grapher to Polar and Dot graphing modes, and to Radian mode. Using $k = 3$, an xy window of $[-12, 24]$ by $[-12, 12]$, $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \pi/48$, graph

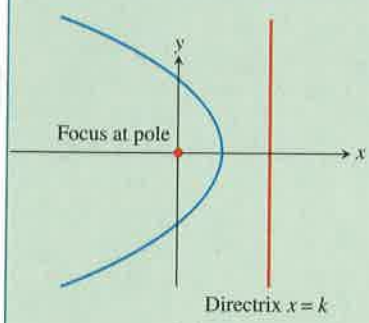
$$r = \frac{ke}{1 + e\cos\theta}$$

for $e = 0.7, 0.8, 1, 1.5, 3$. Identify the type of conic section obtained for each e value. Overlay the five graphs, and explain how changing the value of e affects the graph of $r = ke/(1 + e\cos\theta)$. Explain how the five graphs are similar and how they are different.

Polar Equations for Conics

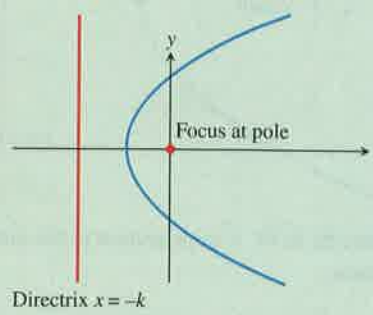
The four standard orientations of a conic in the polar plane are as follows.

(a) $r = \dfrac{ke}{1 + e\cos\theta}$



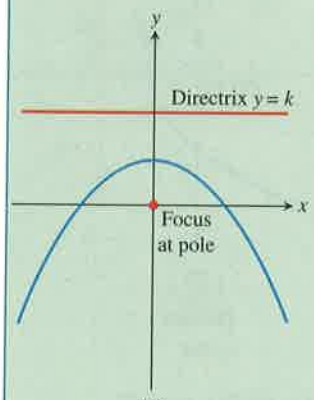
(a)

(b) $r = \dfrac{ke}{1 - e\cos\theta}$



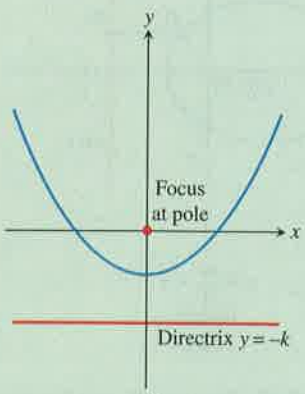
(b)

(c) $r = \dfrac{ke}{1 + e\sin\theta}$



(c)

(d) $r = \dfrac{ke}{1 - e\sin\theta}$



(d)

EXAMPLE 1 Writing and graphing polar equations of conics

Given that the focus is at the pole, write a polar equation for the specified conic, and graph it.

(a) Eccentricity $e = 3/5$, directrix $x = 2$.

(b) Eccentricity $e = 1$, directrix $x = -2$.

(c) Eccentricity $e = 3/2$, directrix $y = 4$.

SOLUTION

(a) Setting $e = 3/5$ and $k = 2$ in $r = \dfrac{ke}{1 + e\cos\theta}$ yields

$$r = \frac{2(3/5)}{1 + (3/5)\cos\theta} = \frac{6}{5 + 3\cos\theta}.$$

Figure 8.43a shows this ellipse and the given directrix.

(b) Setting $e = 1$ and $k = 2$ in $r = \dfrac{ke}{1 - e\cos\theta}$ yields

$$r = \frac{2}{1 - \cos\theta}.$$

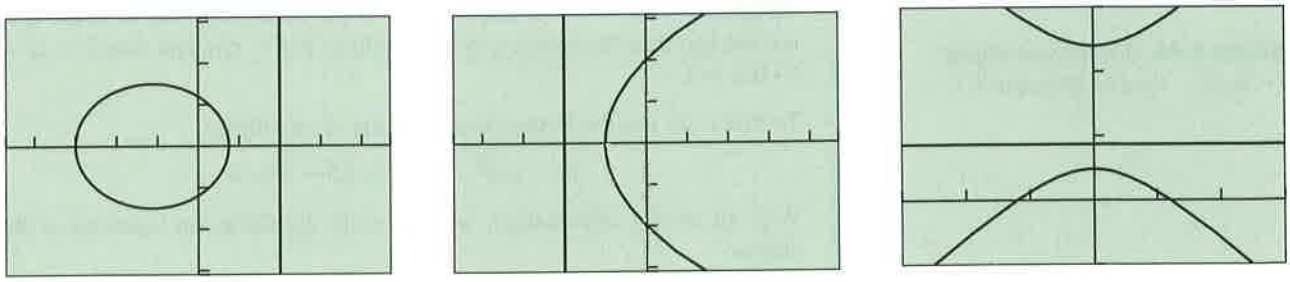
Figure 8.43b shows this parabola and its directrix.

(c) Setting $e = 3/2$ and $k = 4$ in $r = \dfrac{ke}{1 + e\sin\theta}$ yields

$$r = \frac{4(3/2)}{1 + (3/2)\sin\theta} = \frac{12}{2 + 3\sin\theta}.$$

Figure 8.43c shows this hyperbola and the given directrix.

Now try Exercise 1.



[–4.7, 4.7] by [–3.1, 3.1]
(a)

[–4.7, 4.7] by [–3.1, 3.1]
(b)

[–15,15] by [–5, 15]
(c)

FIGURE 8.43 Graphs for Example 1.

Analyzing Polar Equations of Conics

The first step in analyzing the polar equations of a conic section is to use the eccentricity to identify which type of conic the equation represents. Then we determine the equation of the directrix.

EXAMPLE 2 Identifying conics from their polar equations

Determine the eccentricity, the type of conic, and the directrix.

(a) $r = \dfrac{6}{2 + 3\cos\theta}$ **(b)** $r = \dfrac{6}{4 - 3\sin\theta}$

SOLUTION

(a) Dividing numerator and denominator by 2 yields $r = 3/(1 + 1.5\cos\theta)$. So the eccentricity $e = 1.5$, and thus the conic is a hyperbola. The numerator $ke = 3$, so $k = 2$, and thus the equation of the directrix is $x = 2$.

(b) Dividing numerator and denominator by 4 yields $r = 1.5/(1 - 0.75\sin\theta)$. So the eccentricity $e = 0.75$, and thus the conic is an ellipse. The numerator $ke = 1.5$, so $k = 2$, and thus the equation of the directrix is $y = -2$.

Now try Exercise 7.

All of the geometric properties and features of parabolas, ellipses, and hyperbolas developed in Sections 8.1–8.3 still apply in the polar coordinate setting. In Example 3 we use this prior knowledge.

EXAMPLE 3 Analyzing a conic

Analyze the conic section given by the equation $r = 16/(5 - 3\cos\theta)$. Include in the analysis the values of e, a, b, and c.

SOLUTION Dividing numerator and denominator by 5 yields

$$r = \frac{3.2}{1 - 0.6\cos\theta}.$$

So the eccentricity $e = 0.6$, and thus the conic is an ellipse. Figure 8.44 shows this ellipse. The vertices (endpoints of the major axis) have polar coordinates $(8, 0)$ and $(2, \pi)$. So $2a = 8 + 2 = 10$, and thus $a = 5$.

The vertex $(2, \pi)$ is 2 units to the left of the pole, and the pole is a focus of the ellipse. So $a - c = 2$, and thus $c = 3$. An alternative way to find c is to use the fact that the eccentricity of an ellipse is $e = c/a$, and thus $c = ae = 5 \cdot 0.6 = 3$.

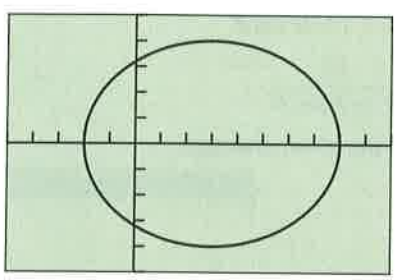
To find b we use the Pythagorean relation of an ellipse:

$$b = \sqrt{a^2 - c^2} = \sqrt{25 - 9} = 4.$$

With all of this information, we can write the Cartesian equation of the ellipse:

$$\frac{(x-3)^2}{25} + \frac{y^2}{16} = 1.$$

Now try Exercise 31.



[–5, 10] by [–5, 5]

FIGURE 8.44 Graph of the ellipse $r = 16/(5 - 3\cos\theta)$. (Example 3)

Orbits Revisited

Polar equations for conics are used extensively in celestial mechanics, the branch of astronomy based on Kepler's work that studies the motion of celestial bodies. The polar equations of conic sections are well suited to the *two-body problem* of celestial mechanics for several reasons. First, the same equations are used for ellipses, parabolas, and hyperbolas—the paths of one body traveling about another. Second, a focus of the conic is always at the pole. This arrangement has two immediate advantages:

- The pole can be thought of as the center of the larger body, such as the Sun, with the smaller body, such as the Earth, following a conic path about the larger body.
- The coordinates given by a polar equation are the distance r between the two bodies and the direction θ from the larger body to the smaller body relative to the axis of the conic path of motion.

For these reasons, polar coordinates are preferred over Cartesian coordinates for studying orbital motion.

TABLE 8.2 SEMIMAJOR AXES AND ECCENTRICITIES OF THE PLANETS

Planet	Semimajor Axis (Gm)	Eccentricity
Mercury	57.9	0.2056
Venus	108.2	0.0068
Earth	149.6	0.0167
Mars	227.9	0.0934
Jupiter	778.3	0.0485
Saturn	1427	0.0560
Uranus	2869	0.0461
Neptune	4497	0.0050
Pluto	5900	0.2484

Source: Shupe, et al., National Geographic Atlas of the World (rev. 6th ed.). Washington, DC: National Geographic Society, 1992, plate 116, and other sources.

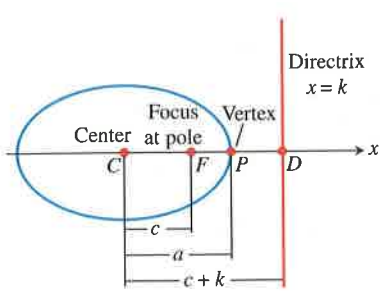


FIGURE 8.45 Geometric relationships within an ellipse.

To use the data in Table 8.2 to create polar equations for the elliptical orbits of the planets, we need to express the equation $r = ke/(1 + e\cos\theta)$ in terms of a and e. We apply the formula $PF = e \cdot PD$ to the ellipse shown in Figure 8.45:

$$e \cdot PD = PF$$

$$e(c + k - a) = a - c \qquad \text{From Figure 8.45}$$

$$e(ae + k - a) = a - ae \qquad \text{Use } e = c/a.$$

$$ae^2 + ke - ae = a - ae \qquad \text{Distribute the } e.$$

$$ae^2 + ke = a \qquad \text{Add } ae.$$

$$ke = a - ae^2 \qquad \text{Subtract } ae^2.$$

$$ke = a(1 - e^2) \qquad \text{Factor.}$$

So the equation $r = ke/(1 + e\cos\theta)$ can be rewritten as follows:

Ellipse with Eccentricity e and Semimajor Axis a

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

In this form of the equation, when $e = 0$, the equation reduces to $r = a$, the equation of a circle with radius a.

EXAMPLE 4 Analyzing a planetary orbit

Find a polar equation for the orbit of Mercury, and use it to approximate its aphelion (farthest distance from the Sun) and perihelion (closest distance to the Sun).

SOLUTION Setting $e = 0.2056$ and $a = 57.9$ in

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \quad \text{yields} \quad r = \frac{57.9(1 - 0.2056^2)}{1 + 0.2056\cos\theta}.$$

Mercury's aphelion is

$$r = \frac{57.9(1 - 0.2056^2)}{1 - 0.2056} \approx 69.8 \text{ Gm}.$$

Mercury's perihelion is

$$r = \frac{57.9(1 - 0.2056^2)}{1 + 0.2056} \approx 46.0 \text{ Gm}.$$

Now try Exercise 41.

QUICK REVIEW 8.5

(For help, go to Sections 6.4.)

In Exercises 1 and 2, solve for r.

1. $(3, \theta) = (r, \theta + \pi)$
2. $(-2, \theta) = (r, \theta + \pi)$

In Exercises 3 and 4, solve for θ.

3. $(1.5, \pi/6) = (-1.5, \theta)$, $-2\pi \le \theta \le 2\pi$
4. $(-3, 4\pi/3) = (3, \theta)$, $-2\pi \le \theta \le 2\pi$

In Exercises 5 and 6, find the focus and directrix of the parabola.

5. $x^2 = 16y$
6. $y^2 = -12x$

In Exercises 7–10, find the foci and vertices of the conic.

7. $\frac{x^2}{9} + \frac{y^2}{4} = 1$
8. $\frac{y^2}{25} + \frac{x^2}{9} = 1$
9. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
10. $\frac{y^2}{36} - \frac{x^2}{4} = 1$

SECTION 8.5 EXERCISES

In Exercises 1–6, find a polar equation for the conic with a focus at the pole and the given eccentricity and directrix. Identify the conic, and graph it.

1. $e = 1, x = -2$ **2.** $e = 5/4, x = 4$

3. $e = 3/5, y = 4$ **4.** $e = 1, y = 2$

5. $e = 7/3, y = -1$ **6.** $e = 2/3, x = -5$

In Exercises 7–14, determine the eccentricity, type of conic, and directrix.

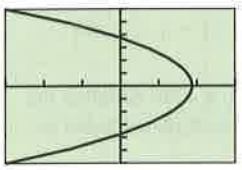
7. $r = \dfrac{2}{1 + \cos\theta}$ **8.** $r = \dfrac{6}{1 + 2\cos\theta}$

9. $r = \dfrac{5}{2 - 2\sin\theta}$ **10.** $r = \dfrac{2}{4 - \cos\theta}$

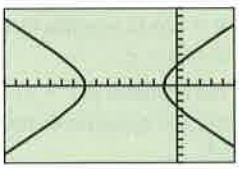
11. $r = \dfrac{20}{6 + 5\sin\theta}$ **12.** $r = \dfrac{42}{2 - 7\sin\theta}$

13. $r = \dfrac{6}{5 + 2\cos\theta}$ **14.** $r = \dfrac{20}{2 + 5\sin\theta}$

In Exercises 15–20, match the polar equation with its graph, and identify the viewing window.



(a)



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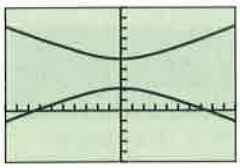

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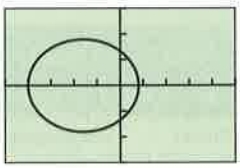
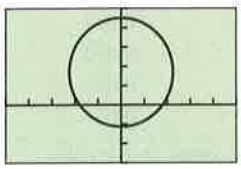
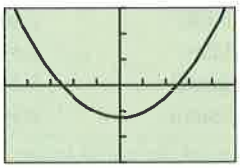


(d)



(e)



(f)

15. $r = \dfrac{8}{3 - 4\cos\theta}$ **16.** $r = \dfrac{4}{3 + 2\cos\theta}$

17. $r = \dfrac{5}{2 - 2\sin\theta}$ **18.** $r = \dfrac{9}{5 - 3\sin\theta}$

19. $r = \dfrac{15}{2 + 5\sin\theta}$ **20.** $r = \dfrac{15}{4 + 4\cos\theta}$

In Exercises 21–24, find a polar equation for the ellipse with a focus at the pole and the given polar coordinates as the endpoints of its major axis.

21. $(1.5, 0)$ and $(6, \pi)$ **22.** $(1.5, 0)$ and $(1, \pi)$

23. $(1, \pi/2)$ and $(3, 3\pi/2)$ **24.** $(3, \pi/2)$ and $(0.75, -\pi/2)$

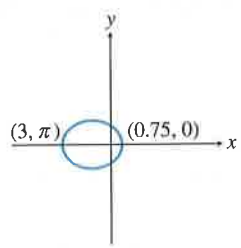
In Exercises 25–28, find a polar equation for the hyperbola with a focus at the pole and the given polar coordinates as the endpoints of its transverse axis.

25. $(3, 0)$ and $(-15, \pi)$ **26.** $(-3, 0)$ and $(1.5, \pi)$

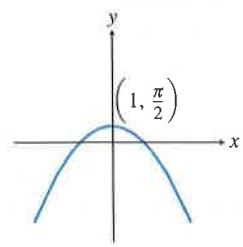
27. $\left(2.4, \dfrac{\pi}{2}\right)$ and $\left(-12, \dfrac{3\pi}{2}\right)$ **28.** $\left(-6, \dfrac{\pi}{2}\right)$ and $\left(2, \dfrac{3\pi}{2}\right)$

In Exercises 29 and 30, find a polar equation for the conic with a focus at the pole.

29.



30.



In Exercises 31–36, graph the conic, and find the values of e, a, b, and c.

31. $r = \dfrac{21}{5 - 2\cos\theta}$ **32.** $r = \dfrac{11}{6 - 5\sin\theta}$

33. $r = \dfrac{24}{4 + 2\sin\theta}$ **34.** $r = \dfrac{16}{5 + 3\cos\theta}$

35. $r = \dfrac{16}{3 + 5\cos\theta}$ **36.** $r = \dfrac{12}{1 - 5\sin\theta}$

In Exercises 37 and 38, determine a Cartesian equation for the given polar equation.

37. $r = \dfrac{4}{2 - \sin\theta}$ **38.** $r = \dfrac{6}{1 + 2\cos\theta}$

In Exercises 39 and 40, use the fact that $k = 2p$ is twice the focal length and half the focal width, to determine a Cartesian equation of the parabola whose polar equation is given.

39. $r = \dfrac{4}{2 - 2\cos\theta}$ **40.** $r = \dfrac{12}{3 + 3\cos\theta}$

41. Halley's Comet The orbit of Halley's comet has a semimajor axis of 18.09 AU and an orbital eccentricity of 0.97. Compute its perihelion and aphelion distances.

42. Uranus The orbit of the planet Uranus has a semimajor axis of 19.18 AU and an orbital eccentricity of 0.0461. Compute its perihelion and aphelion distances.

In Exercises 43 and 44, the velocity of an object traveling in a circular orbit of radius r (distance from center of planet in meters) around a planet is given by

$$v = \sqrt{\frac{3.99 \times 10^{14}\, k}{r}} \text{ m/sec},$$

where k is a constant related to the mass of the planet and the orbiting object.

43. Group Activity Lunar Module A lunar excursion module is in a circular orbit 250 km above the surface of the Moon. Assume that the Moon's radius is 1740 km and that $k = 0.012$. Find the following.

(a) the velocity of the lunar module

(b) the length of time required for the lunar module to circle the moon once

44. Group Activity Mars Satellite A satellite is in a circular orbit 1000 mi above Mars. Assume that the radius of Mars is 2100 mi and that $k = 0.11$. Find the velocity of the satellite.

Standardized Test Questions

45. True or False The equation $r = ke/(1 + e \cos \theta)$ yields no true circles. Justify your answer.

46. True or False The equation $r = a(1 - e^2)/(1 + e \cos \theta)$ yields no true parabolas. Justify your answer.

In Exercises 47–50, solve the problem without using a calculator.

47. Multiple Choice Which ratio of distances is constant for a point on a nondegenerate conic?

(a) distance to center : distance to directrix

(b) distance to focus : distance to vertex

(c) distance to vertex : distance to directrix

(d) distance to focus : distance to directrix

(e) distance to center : distance to vertex

48. Multiple Choice Which type of conic section has an eccentricity greater than one?

(a) an ellipse

(b) a parabola

(c) a hyperbola

(d) two parallel lines

(e) a circle

49. Multiple Choice For a conic expressed by $r = ke/(1 + e \sin \theta)$, which point is located at the pole?

(a) the center

(b) a focus

(c) a vertex

(d) an endpoint of the minor axis

(e) an endpoint of the conjugate axis

50. Multiple Choice Which of the following is **not** a polar equation of a conic?

(a) $r = 1 + 2 \cos \theta$

(b) $r = 1/(1 + \sin \theta)$

(c) $r = 3$

(d) $r = 1/(2 - \cos \theta)$

(e) $r = 1/(1+2 \cos \theta)$

Explorations

51. Planetary Orbits Use the polar equation $r = a(1 - e^2)/(1 + e \cos \theta)$ in completing the following activities.

(a) Use the fact that $-1 \le \cos \theta \le 1$ to prove that the perihelion distance of any planet is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

(b) Use $e = c/a$ to confirm that $a(1 - e) = a - c$ and $a(1 + e) = a + c$.

(c) Use the formulas $a(1 - e)$ and $a(1 + e)$ to compute the perihelion and aphelion distances of each planet listed in Table 8.3.

(d) For which of these planets is the difference between the perihelion and aphelion distance the greatest?

TABLE 8.3 SEMIMAJOR AXES AND ECCENTRICITIES OF THE SIX INNERMOST PLANETS

Planet	Semimajor Axis (AU)	Eccentricity
Mercury	0.3871	0.206
Venus	0.7233	0.007
Earth	1.0000	0.017
Mars	1.5237	0.093
Jupiter	5.2026	0.048
Saturn	9.5547	0.056

Source: Encrenaz & Bibring. The Solar System (2nd ed.). New York: Springer, p. 5.

52. Using the Astronomer's Equation for Conics Using Dot mode, $a = 2$, an xy window of $[-13, 5]$ by $[-6, 6]$, $\theta\text{min} = 0$, $\theta\text{max} = 2\pi$, and $\theta\text{step} = \pi/48$, graph $r = a\,(1 - e^2)/(1 + e \cos \theta)$ for $e = 0, 0.3, 0.7, 1.5, 3$. Identify the type of conic section obtained for each e value. What happens when $e = 1$?

Extending the Ideas

53. Revisiting Figure 8.41 In Figure 8.41, if $r < 0$ or $r \cos \theta > k$, then we must use $PD = |k - r \cos \theta|$ and $PF = |r|$. Prove that, even in these cases, the resulting equation is still $r = ke/(1 + e \cos \theta)$.

54. Deriving Other Polar Forms for Conics Using Figure 8.41 as a guide, draw an appropriate diagram for and derive the equation.

(a) $r = \dfrac{ke}{1 - e \cos \theta}$

(b) $r = \dfrac{ke}{1 + e \sin \theta}$

(c) $r = \dfrac{ke}{1 - e \sin \theta}$

55. Focal Widths Using polar equations, derive formulas for the focal width of an ellipse and the focal width of a hyperbola. Begin by defining focal width for these conics in a manner analogous to the definition of the focal width of a parabola given in Section 8.1.

56. Prove that for a hyperbola the formula $r = ke/(1 - e \cos \theta)$ is equivalent to $r = a(e^2 - 1)/(1 - e \cos \theta)$, where a is the semitransverse axis of the hyperbola.

57. Connecting Polar to Rectangular Consider the ellipse

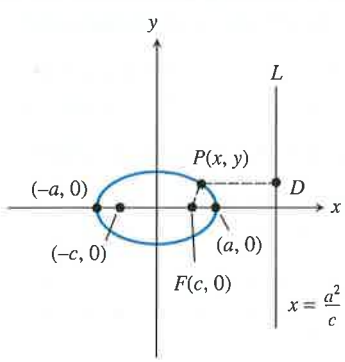
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where half the length of the major axis is a, and the foci are $(\pm c, 0)$ such that $c^2 = a^2 - b^2$. Let L be the vertical line $x = a^2/c$.

(a) Prove that L is a directrix for the ellipse. (*Hint*: Prove that PF/PD is the constant c/a, where P is a point on the ellipse, and D is the point on L such that PD is perpendicular to L.)

(b) Prove that the eccentricity is $e = c/a$.

(c) Prove that the distance from F to L is $a/e - ea$.



58. Connecting Polar to Rectangular Consider the hyperbola

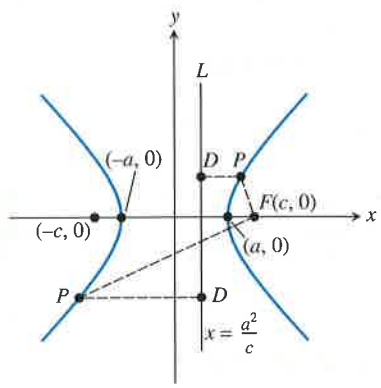
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where half the length of the transverse axis is a, and the foci are $(\pm c, 0)$ such that $c^2 = a^2 + b^2$. Let L be the vertical line $x = a^2/c$.

(a) Prove that L is a directrix for the hyperbola. (*Hint*: Prove that PF/PD is the constant c/a, where P is a point on the hyperbola, and D is the point on L such that PD is perpendicular to L.)

(b) Prove that the eccentricity is $e = c/a$.

(c) Prove that the distance from F to L is $ea - a/e$



8.6 THREE-DIMENSIONAL CARTESIAN COORDINATE SYSTEM

What you'll learn about

- Three-Dimensional Cartesian Coordinates
- Distance and Midpoint Formulas
- Equation of a Sphere
- Planes and Other Surfaces
- Vectors in Space
- Lines in Space

. . . and why

This is the analytic geometry of our physical world.

Three-Dimensional Cartesian Coordinates

In Sections P.2 and P.4, we studied Cartesian coordinates and the associated basic formulas and equations for the two-dimensional plane; we now extend these ideas to *three-dimensional space*. In the plane, we used two axes and ordered pairs to name points; in space, we use three mutually perpendicular axes and ordered triples of real numbers to name points. See Figure 8.46.

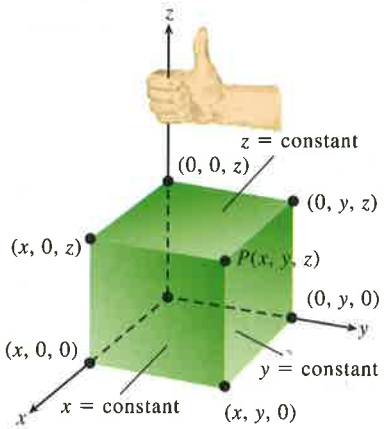


FIGURE 8.46 The point $P(x, y, z)$ in Cartesian space.

Notice that Figure 8.46 exhibits several important features of the *three-dimensional Cartesian coordinate system*:

- The axes are labeled x, y, and z, and these three **coordinate axes** form a **right-handed coordinate frame**: When you hold your right hand with fingers curving from the positive x-axis toward the positive y-axis, your thumb points in the direction of the positive z-axis.
- A point P in space is uniquely paired with an ordered triple (x, y, z) of real numbers. The numbers x, y, and z are the **Cartesian coordinates of P**.
- Points on the axes have the form $(x, 0, 0)$, $(0, y, 0)$, or $(0, 0, z)$, with $(x, 0, 0)$ on the x-axis, $(0, y, 0)$ on the y-axis, and $(0, 0, z)$ on the z-axis.

In Figure 8.47, the axes are paired to determine the **coordinate planes**:

- The coordinate planes are the **xy-plane**, the **xz-plane**, and the **yz-plane**, and have equations $z = 0$, $y = 0$, and $x = 0$, respectively.
- Points on the coordinate planes have the form $(x, y, 0)$, $(x, 0, z)$, or $(0, y, z)$, with $(x, y, 0)$ on the xy-plane, $(x, 0, z)$ on the xz-plane, and $(0, y, z)$ on the yz-plane.

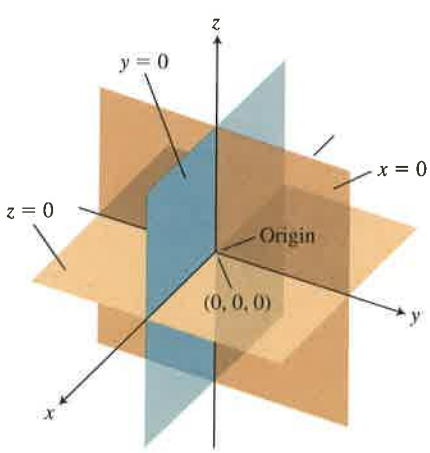


FIGURE 8.47 The coordinate planes divide space into eight octants.

- The coordinate planes meet at the **origin**, (0, 0, 0).
- The coordinate planes divide space into eight regions called **octants**, with the **first octant** containing all points in space with three positive coordinates.

EXAMPLE 1 Locating a point in Cartesian space

Draw a sketch that shows the point (2, 3, 5).

SOLUTION To locate the point (2, 3, 5), we first sketch a right-handed three-dimensional coordinate frame. We then draw the planes $x = 2$, $y = 3$, and $z = 5$, which parallel the coordinate planes $x = 0$, $y = 0$, and $z = 0$, respectively. The point (2, 3, 5) lies at the intersection of the planes $x = 2$, $y = 3$, and $z = 5$, as shown in Figure 8.48. Now try Exercise 1.

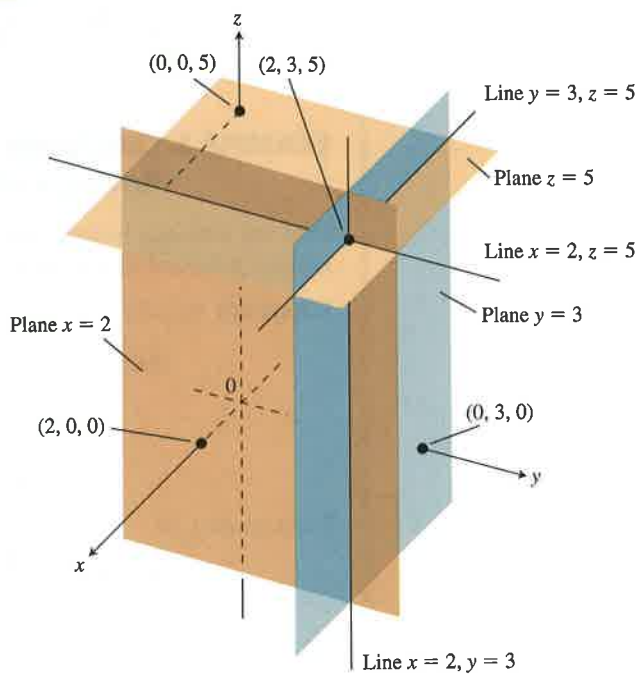


FIGURE 8.48 The planes $x = 2$, $y = 3$, and $z = 5$ determine the point (2, 3, 5). (Example 1)

Distance and Midpoint Formulas

The distance and midpoint formulas for space are natural generalizations of the corresponding formulas for the plane.

Distance Formula (Cartesian Space)

The distance $d(P, Q)$ between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space is

$$d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Just as in the plane, the coordinates of the midpoint of a line segment are the averages for the coordinates of the endpoints of the segment.

Midpoint Formula (Cartesian Space)

The midpoint M of the line segment PQ with endpoints $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

EXAMPLE 2 Calculating a distance and finding a midpoint

Find the distance between the points $P(-2, 3, 1)$ and $Q(4, -1, 5)$, and find the midpoint of line segment PQ.

SOLUTION The distance is given by

$$\begin{aligned} d(P, Q) &= \sqrt{(-2 - 4)^2 + (3 + 1)^2 + (1 - 5)^2} \\ &= \sqrt{36 + 16 + 16} \\ &= \sqrt{68} \approx 8.25 \end{aligned}$$

The midpoint is

$$M = \left(\frac{-2 + 4}{2}, \frac{3 - 1}{2}, \frac{1 + 5}{2}\right) = (1, 1, 3).$$

Now try Exercises 5 and 9.

Equation of a Sphere

A *sphere* is the three-dimensional analogue of a circle: In space, the set of points that lie a fixed distance from a fixed point is a **sphere**. The fixed distance is the **radius**, and the fixed point is the **center** of the sphere. The point $P(x, y, z)$ is a point of the sphere with center (h, k, l) and radius r if and only if

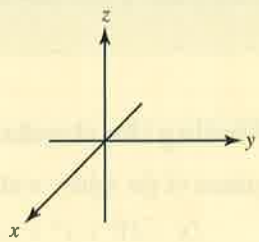
$$\sqrt{(x - h)^2 + (y - k)^2 + (z - l)^2} = r.$$

Squaring both sides gives the standard equation shown on page 690.

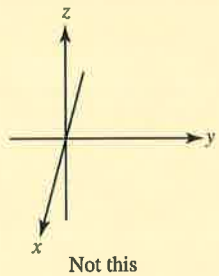
Drawing Lesson

How to Draw Three-Dimensional Objects to Look Three-Dimensional

1. Make the angle between the positive x-axis and the positive y-axis large enough.

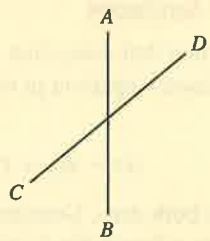


This

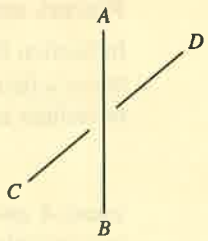


Not this

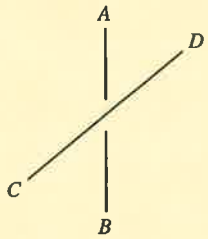
2. Break lines. When one line passes behind another, break it to show that it doesn't touch and that part of it is hidden.



Intersecting

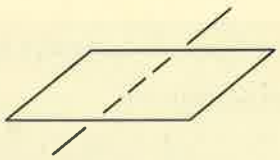


CD behind *AB*

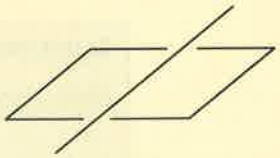


AB behind *CD*

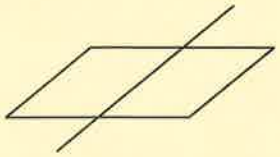
3. Dash or omit hidden portions of lines. Don't let the line touch the boundary of the parallelogram that represents the plane, unless the line lies in the plane.



Line below plane

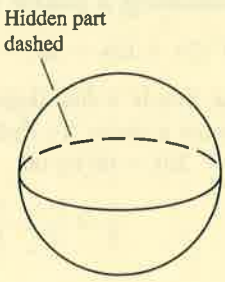


Line above plane

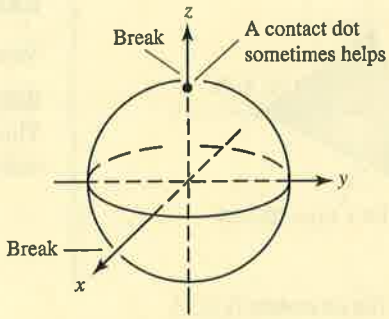


Line *in* plane

4. Spheres: Draw the sphere first (outline and equator); draw axes, if any, later. Use line breaks and dashed lines.



Sphere first



Axes later

Standard Equation of a Sphere

A point $P(x, y, z)$ is on the sphere with center (h, k, l) and radius r if and only if

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2.$$

EXAMPLE 3 Finding the standard equation of a sphere

The standard equation of the sphere with center $(2, 0, -3)$ and radius 7 is

$$(x-2)^2 + y^2 + (z+3)^2 = 49.$$

Now try Exercises 13.

Planes and Other Surfaces

In Section P.4, we learned that every line in the Cartesian plane can be written as a first-degree (linear) equation in two variables; that is, every line can be written as

$$Ax + By + C = 0,$$

where A and B are not both zero. Conversely, every first-degree equation in two variables represents a line in the Cartesian plane.

In an analogous way, every **plane** in Cartesian space can be written as a **first-degree equation in three variables**:

Equation for a Plane in Cartesian Space

Every plane can be written as

$$Ax + By + Cz + D = 0,$$

where A, B, and C are not all zero. Conversely, every first-degree equation in three variables represents a plane in Cartesian space.

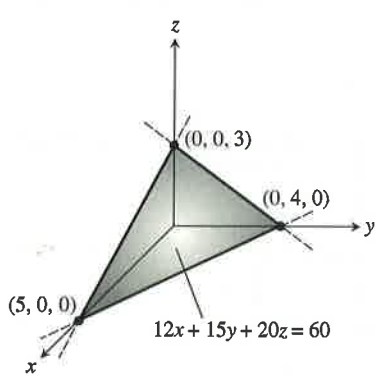


FIGURE 8.49 The Intercepts (5, 0, 0), (0, 4, 0), and (0, 0, 3) determine the plane $12x + 15y + 20z = 60$. (Example 4)

EXAMPLE 4 Sketching a plane in space

Sketch the graph of $12x + 15y + 20z = 60$.

SOLUTION Because this is a first-degree equation, its graph is a plane. Three points determine a plane. To find three points, we first divide both sides of $12x + 15y + 20z = 60$ by 60:

$$\frac{x}{5} + \frac{y}{4} + \frac{z}{3} = 1.$$

In this form, it is easy to see that the points $(5, 0, 0)$, $(0, 4, 0)$, and $(0, 0, 3)$ satisfy the equation. These are the points where the graph crosses the coordinate axes. Figure 8.49 shows the completed sketch.

Now try Exercise 17.

Equations in the three variables x, y, and z generally graph as surfaces in three-dimensional space. Just as in the plane, second-degree equations are of particular interest. Recall that second-degree equations in two variables yield conic sections in the Cartesian plane. In space, second-degree equations in *three* variables yield **quadric surfaces**: The paraboloids, ellipsoids, and hyperboloids of revolution that have special reflective properties are all quadric surfaces, as are such exotic-sounding surfaces as hyperbolic paraboloids and elliptic hyperboloids.

Other surfaces of interest include graphs of **functions of two variables**, whose equations have the form $z = f(x, y)$. Some examples are $z = x \ln y$, $z = \sin(xy)$, and $z = \sqrt{1 - x^2 - y^2}$. The last equation graphs as a hemisphere (see Exercise 63). Equations of the form $z = f(x, y)$ can be graphed using some graphing calculators and most computer algebra software. Quadric surfaces and functions of two variables are studied in most university-level calculus course sequences.

Vectors in Space

In space, just as in the plane, the sets of equivalent directed line segments are called *vectors*. They are used to represent forces, displacements, and velocities in three dimensions. In space, we use ordered triples to denote vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle.$$

The **zero vector** is $\mathbf{0} = \langle 0, 0, 0 \rangle$, and the **standard unit vectors** are $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$. As shown in Figure 8.50, the vector $\mathbf{v}$ can be expressed in terms of these standard unit vectors:

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

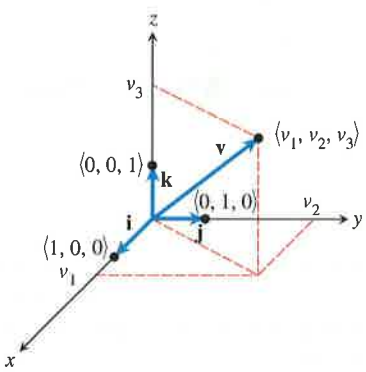


FIGURE 8.50 The vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.

The vector $\mathbf{v}$ that is represented by the directed line segment $\overrightarrow{PQ}$ from $P(a, b, c)$ to $Q(x, y, z)$ is

$$\mathbf{v} = \overrightarrow{PQ} = \langle x - a, y - b, z - c \rangle = (x - a)\mathbf{i} + (y - b)\mathbf{j} + (z - c)\mathbf{k}.$$

A vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be multiplied by a scalar (real number) c as follows:

$$c\mathbf{v} = c\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle.$$

Many other properties of vectors generalize in a natural way when we move from two to three dimensions:

Vector Relationships in Space

For vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$,

- **Equality:** $\mathbf{v} = \mathbf{w}$ if and only if $v_1 = w_1$, $v_2 = w_2$, and $v_3 = w_3$
- **Addition:** $\mathbf{v} + \mathbf{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$
- **Subtraction:** $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$
- **Magnitude:** $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- **Dot product:** $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$
- **Unit vector:** $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$, $\mathbf{v} \neq \mathbf{0}$, is the unit vector in the direction of $\mathbf{v}$.

EXAMPLE 5 Computing with vectors

(a) $3\langle -2, 1, 4 \rangle = \langle 3 \cdot -2, 3 \cdot 1, 3 \cdot 4 \rangle = \langle -6, 3, 12 \rangle$

(b) $\langle 0, 6, -7 \rangle + \langle -5, 5, 8 \rangle = \langle 0 - 5, 6 + 5, -7 + 8 \rangle = \langle -5, 11, 1 \rangle$

(c) $\langle 1, -3, 4 \rangle - \langle -2, -4, 5 \rangle = \langle 1 + 2, -3 + 4, 4 - 5 \rangle = \langle 3, 1, -1 \rangle$

(d) $|\langle 2, 0, -6 \rangle| = \sqrt{2^2 + 0^2 + 6^2} = \sqrt{40} \approx 6.32$

(e) $\langle 5, 3, -1 \rangle \cdot \langle -6, 2, 3 \rangle = 5 \cdot (-6) + 3 \cdot 2 + (-1) \cdot 3$

$$= -30 + 6 - 3 = -27$$

Now try Exercises 23–26.

EXAMPLE 6 Using vectors in space

A jet airplane just after takeoff is pointed due east. Its air velocity vector makes an angle of 30° with flat ground with an airspeed of 250 mph. If the wind is out of the southeast at 32 mph, calculate a vector that represents the plane's velocity relative to the point of takeoff.

SOLUTION Let $\mathbf{i}$ point east, $\mathbf{j}$ point north, and $\mathbf{k}$ point up. The plane's air velocity is

$$\mathbf{a} = \langle 250 \cos 30°, 0, 250 \sin 30° \rangle \approx \langle 216.506, 0, 125 \rangle,$$

and the wind velocity, which is pointing northwest, is

$$\mathbf{w} = \langle 32 \cos 135°, 32 \sin 135°, 0 \rangle \approx \langle -22.627, 22.627, 0 \rangle.$$

The velocity relative to the ground is $\mathbf{v} = \mathbf{a} + \mathbf{w}$, so

$$\begin{aligned} \mathbf{v} &\approx \langle 216.506, 0, 125 \rangle + \langle -22.627, 22.627, 0 \rangle \\ &\approx \langle 193.88, 22.63, 125 \rangle \\ &= 193.88\mathbf{i} + 22.63\mathbf{j} + 125\mathbf{k} \end{aligned}$$

Now try Exercise 33.

In Exercise 64, you will be asked to interpret the meaning of the velocity vector obtained in Example 6.

Lines in Space

We have seen that first-degree equations in three variables graph as planes in space. So how do we get lines? There are several ways. First notice that to specify the x-axis, which is a line, we could use the pair of first-degree equations $y = 0$ and $z = 0$. As alternatives to using a pair of Cartesian equations, we can specify any line in space using

- one vector equation, or
- a set of three parametric equations.

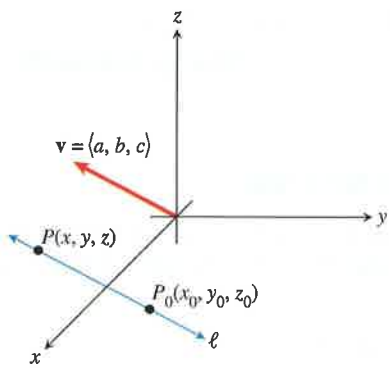


FIGURE 8.51 The line ℓ is parallel to the direction vector $\mathbf{v} = \langle a, b, c\rangle$.

Suppose ℓ is a line through the point $P_0(x_0, y_0, z_0)$ and in the direction of a nonzero vector $\mathbf{v} = \langle a, b, c\rangle$ (Figure 8.51). Then for any point $P(x, y, z)$ on ℓ,

$$\overrightarrow{P_0P} = t\mathbf{v}$$

for some real number t. The vector $\mathbf{v}$ is a **direction vector** for line ℓ. If $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \overrightarrow{OP_0} = \langle x_0, y_0, z_0\rangle$, then $\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}$. So the vector equation of the line ℓ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Writing this in component form yields

$$\begin{aligned}\langle x, y, z\rangle &= \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct\rangle,\end{aligned}$$

which can be expressed as the parametric equations

$$x = x_0 + at, \quad y = y_0 + bt, \quad \text{and} \quad z = z_0 + ct.$$

Equations for a Line in Space

If ℓ is a line through the point $P_0(x_0, y_0, z_0)$ in the direction of a nonzero vector $\mathbf{v} = \langle a, b, c\rangle$, then a point $P(x, y, z)$ is on ℓ if and only if

- **Vector form:** $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$; or
- **Parametric form:** $x = x_0 + at$, $y = y_0 + bt$, and $z = z_0 + ct$,

where t is a real number.

EXAMPLE 7 Finding equations for a line

The line through $P_0(4, 3, -1)$ with direction vector $\mathbf{v} = \langle -2, 2, 7\rangle$ can be written

- in vector form as $\mathbf{r} = \langle 4, 3, -1\rangle + t\langle -2, 2, 7\rangle$; or
- in parametric form as $x = 4 - 2t$, $y = 3 + 2t$, and $z = -1 + 7t$.

Now try Exercise 35.

EXAMPLE 8 Finding equations for a line

Using the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, write a vector equation for the line containing the points $A(3, 0, -2)$ and $B(-1, 2, -5)$, and compare it to the parametric equations for the line.

SOLUTION The line is in the direction of

$$\mathbf{v} = \overrightarrow{AB} = \langle -1 - 3, 2 - 0, -5 + 2\rangle = \langle -4, 2, -3\rangle.$$

So using $\mathbf{r}_0 = \overrightarrow{OA}$, the vector equation of the line becomes:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

$$\langle x, y, z\rangle = \langle 3, 0, -2\rangle + t\langle -4, 2, -3\rangle$$

$$\langle x, y, z\rangle = \langle 3 - 4t, 2t, -2 - 3t\rangle$$

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (3 - 4t)\mathbf{i} + 2t\mathbf{j} + (-2 - 3t)\mathbf{k}$$

The parametric equations are the three component equations

$$x = 3 - 4t,\ y = 2t, \text{ and } z = -2 - 3t.$$

Now try Exercise 41.

QUICK REVIEW 8.6

(For help, go to Sections 6.1 and 6.3.)

In Exercises 1–3, let $P(x, y)$ and $Q(2, -3)$ be points in the xy-plane.

1. Compute the distance between P and Q.
2. Find the midpoint of the line segment PQ.
3. If P is 5 units from Q, describe the position of P.

In Exercises 4–6, let $\mathbf{v} = \langle -4, 5\rangle = -4\mathbf{i} + 5\mathbf{j}$ be a vector in the xy-plane.

4. Find the magnitude of $\mathbf{v}$.
5. Find a unit vector in the direction of $\mathbf{v}$.
6. Find a vector 7 units long in the direction of $-\mathbf{v}$.
7. Give a geometric description of the graph of $(x + 1)^2 + (y - 5)^2 = 25$ in the xy-plane.
8. Give a geometric description of the graph of $x = 2 - t$, $y = -4 + 2t$ in the xy-plane.
9. Find the center and radius of the circle $x^2 + y^2 + 2x - 6y + 6 = 0$ in the xy-plane.
10. Find a vector from $P(2, 5)$ to $Q(-1, -4)$ in the xy-plane.

SECTION 8.6 EXERCISES

In Exercises 1–4, draw a sketch that shows the point.

1. $(3, 4, 2)$ **2.** $(2, -3, 6)$

3. $(1, -2, -4)$ **4.** $(-2, 3, -5)$

In Exercises 5–8, compute the distance between the points.

5. $(-1, 2, 5), (3, -4, 6)$

6. $(2, -1, -8), (6, -3, 4)$

7. $(a, b, c), (1, -3, 2)$

8. $(x, y, z), (p, q, r)$

In Exercises 9–12, find the midpoint of the segment PQ.

9. $P(-1, 2, 5), Q(3, -4, 6)$

10. $P(2, -1, -8), Q(6, -3, 4)$

11. $P(2x, 2y, 2z), Q(-2, 8, 6)$

12. $P(-a, -b, -c), Q(3a, 3b, 3c)$

In Exercises 13–16, write an equation for the sphere with the given point as its center and the given number as its radius.

13. $(5, -1, -2), 8$ **14.** $(-1, 5, 8), \sqrt{5}$

15. $(1, -3, 2), \sqrt{a}, a > 0$ **16.** $(p, q, r), 6$

In Exercises 17–22, sketch a graph of the equation. Label all intercepts.

17. $x + y + 3z = 9$ **18.** $x + y - 2z = 8$

19. $x + z = 3$ **20.** $2y + z = 6$

21. $x - 3y = 6$ **22.** $x = 3$

In Exercises 23–32, evaluate the expression using $\mathbf{r} = \langle 1, 0, -3\rangle$, $\mathbf{v} = \langle -3, 4, -5\rangle$, and $\mathbf{w} = \langle 4, -3, 12\rangle$.

23. $\mathbf{r} + \mathbf{v}$ **24.** $\mathbf{r} - \mathbf{w}$

25. $\mathbf{v} \cdot \mathbf{w}$ **26.** $|\mathbf{w}|$

27. $\mathbf{r} \cdot (\mathbf{v} + \mathbf{w})$ **28.** $(\mathbf{r} \cdot \mathbf{v}) + (\mathbf{r} \cdot \mathbf{w})$

29. $\mathbf{w}/|\mathbf{w}|$ **30.** $\mathbf{i} \cdot \mathbf{r}$

31. $\langle \mathbf{i} \cdot \mathbf{v}, \mathbf{j} \cdot \mathbf{v}, \mathbf{k} \cdot \mathbf{v}\rangle$ **32.** $(\mathbf{r} \cdot \mathbf{v})\mathbf{w}$

In Exercises 33 and 34, let $\mathbf{i}$ point east, $\mathbf{j}$ point north, and $\mathbf{k}$ point up.

33. **Three-Dimensional Velocity** An airplane just after takeoff is headed west and is climbing at a 20° angle relative to flat ground with an airspeed of 200 mph. If the wind is out of the northeast at 10 mph, calculate a vector $\mathbf{v}$ that represents the plane's velocity relative to the point of takeoff.

34. **Three-Dimensional Velocity** A rocket soon after takeoff is headed east and is climbing at a 80° angle relative to flat ground with an airspeed of 12,000 mph. If the wind is out of the southwest at 8 mph, calculate a vector $\mathbf{v}$ that represents the rocket's velocity relative to the point of takeoff.

In Exercises 35–38, write the vector and parametric forms of the line through the point P_0 in the direction of $\mathbf{v}$.

35. $P_0(2, -1, 5), \mathbf{v} = \langle 3, 2, -7\rangle$

36. $P_0(-3, 8, -1), \mathbf{v} = \langle -3, 5, 2\rangle$

37. $P_0(6, -9, 0), \mathbf{v} = \langle 1, 0, -4\rangle$

38. $P_0(0, -1, 4), \mathbf{v} = \langle 0, 0, 1\rangle$

In Exercises 39–48, use the points $A(-1, 2, 4)$, $B(0, 6, -3)$, and $C(2, -4, 1)$.

39. Find the distance from A to the midpoint of BC.

40. Find the vector from A to the midpoint of BC.

41. Write a vector equation of the line through A and B.

42. Write a vector equation of the line through A and the midpoint of BC.

43. Write parametric equations for the line through A and C.

44. Write parametric equations for the line through B and C.

45. Write parametric equations for the line through B and the midpoint of AC.

46. Write parametric equations for the line through C and the midpoint of AB.

47. Is ΔABC equilateral, isosceles, or scalene?

48. If M is the midpoint of BC, what is the midpoint of AM?

In Exercises 49–52, **(a)** sketch the line defined by the pair of equations, and **(b)** **Writing to Learn** give a geometric description of the line, including its direction and its position relative to the coordinate frame.

49. $x = 0, y = 0$ **50.** $x = 0, z = 2$

51. $x = -3, y = 0$ **52.** $y = 1, z = 3$

53. Write a vector equation for the line through the distinct points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

54. Write parametric equations for the line through the distinct points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

55. Generalizing the Distance Formula Prove that the distance $d(P, Q)$ between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in space is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$ by using the point $R(x_2, y_2, z_1)$, the two-dimensional distance formula within the plane $z = z_1$, the one-dimensional distance formula within the line $\mathbf{r} = \langle x_2, y_2, t\rangle$, and the Pythagorean theorem. [*Hint:* A sketch may help you visualize the situation.]

56. Generalizing a Property of the Dot Product Prove $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$ where $\mathbf{u}$ is a vector in three-dimensional space.

Standardized Test Questions

57. True or False $x^2 + 4y^2 = 1$ represents a surface in space. Justify your answer.

58. True or False The parametric equation $x = 1 + 0t$, $y = 2 - 0t$, $z = -5 + 0t$ represent a line in space. Justify your answer.

In Exercises 59–62, solve the problem without using a calculator.

59. Multiple Choice A first-degree equation in three variables graphs as

(a) a line.

(b) a plane.

(c) a sphere.

(d) a paraboloid.

(e) an ellipsoid.

60. Multiple Choice Which of the following is **not** a quadric surface?

(a) a plane

(b) a sphere

(c) an ellipsoid

(d) an elliptic paraboloid

(e) a hyperbolic paraboloid

61. Multiple Choice If $\mathbf{v}$ and $\mathbf{w}$ are vectors and c is a scalar, which of these is **a scalar**?

(a) $\mathbf{v} + \mathbf{w}$

(b) $\mathbf{v} - \mathbf{w}$

(c) $\mathbf{v} \cdot \mathbf{w}$

(d) $c\mathbf{v}$

(e) $|\mathbf{v}|\mathbf{w}$

62. Multiple Choice The parametric form of the line $\mathbf{r} = \langle 2, -3, 0\rangle + t\langle 1, 0, -1\rangle$ is

(a) $x = 2 - 3t$, $y = 0 + 1t$, $z = 0 - 1t$.

(b) $x = 2t$, $y = -3 + 0t$, $z = 0 - 1t$.

(c) $x = 1 + 2t$, $y = 0 - 3t$, $z = -1 + 0t$.

(d) $x = 1 + 2t$, $y = -3$, $z = -1t$.

(e) $x = 2 + t$, $y = -3$, $z = -t$.

Explorations

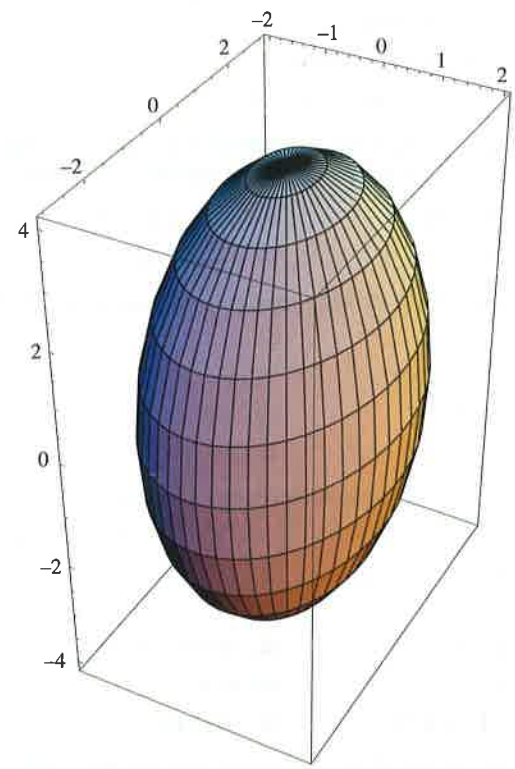
63. Group Activity Writing to Learn The figure shows a graph of the ellipsoid $x^2/9 + y^2/4 + z^2/16 = 1$ drawn in a *box* using *Mathematica* computer software.

(a) Describe its cross sections in each of the three coordinate planes, that is, for $z = 0$, $y = 0$, and $x = 0$. In your description, include the name of each cross section and its position relative to the coordinate frame.

(b) Explain algebraically why the graph of $z = \sqrt{1 - x^2 - y^2}$ is half of a sphere. What is the equation of the related whole sphere?

(c) By hand, sketch the graph of the hemisphere $z = \sqrt{1 - x^2 - y^2}$. Check your sketch using a 3D grapher if you have access to one.

(d) Explain how the graph of an ellipsoid is related to the graph of a sphere and why a sphere is a *degenerate* ellipsoid.



64. Revisiting Example 6 Read Example 6. Then using $\mathbf{v} = 193.88\mathbf{i} + 22.63\mathbf{j} + 125\mathbf{k}$, establish the following:

(a) The plane's compass bearing is 83.34°.

(b) Its speed downrange (that is, ignoring the vertical component) is 195.2 mph.

(c) The plane is climbing at an angle of 32.63°.

(d) The plane's overall speed is 231.8 mph.

Extending the Ideas

The **cross product $\mathbf{u} \times \mathbf{v}$** of the vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

Use this definition in Exercises 65–68.

65. $\langle 1, -2, 3\rangle \times \langle -2, 1, -1\rangle$

66. $\langle 4, -1, 2\rangle \times \langle 1, -3, 2\rangle$

67. Prove that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

68. Assuming the theorem about angles between vectors (Section 6.2, p. 515) holds for three-dimensional vectors, prove that $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ if they are nonzero.

CHAPTER 8 Key Ideas

PROPERTIES, THEOREMS, AND FORMULAS

Parabolas with Vertex (h, k) (p. 635)
Ellipses with Center (h, k) (p. 646)
Hyperbolas with Center (h, k) (p. 658)
Translation Formulas (p. 668)
Rotation Formulas (p. 669)
Discriminant Test (p. 672)
Focus–Directrix–Eccentricity Relationship (p. 676)
Distance Formula (Cartesian Space) (p. 688)
Midpoint Formula (Cartesian Space) (p. 688)
Standard Equation of a Sphere (p. 690)
Vector Relationships in Space (p. 692)
Equations for a Line in Space (p. 693)

PROCEDURES

How to Sketch the Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (p. 644)
How to Sketch the Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (p. 656)
How to Draw Three-Dimensional Objects to Look Three-Dimensional (p. 689)

CHAPTER 8 Review Exercises

The collection of exercises marked in red could be used as a chapter test.

In Exercises 1–4, find the vertex, focus, directrix, and focal width of the parabola, and sketch the graph.

1. $y^2 = 12x$

2. $x^2 = -8y$

3. $(x + 2)^2 = -4(y - 1)$

4. $(y + 2)^2 = 16x$

In Exercises 5–12, identify the type of conic. Find the center, vertices, and foci of the conic, and sketch its graph.

5. $\frac{y^2}{8} + \frac{x^2}{5} = 1$

6. $\frac{y^2}{16} - \frac{x^2}{49} = 1$

7. $\frac{x^2}{25} - \frac{y^2}{36} = 1$

8. $\frac{x^2}{49} - \frac{y^2}{9} = 1$

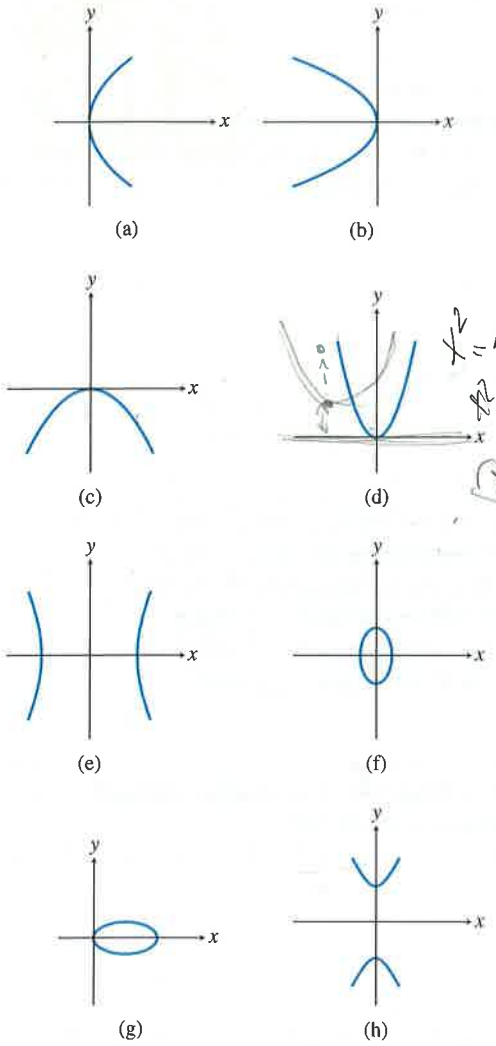
9. $\frac{(x + 3)^2}{18} - \frac{(y - 5)^2}{28} = 1$

10. $\frac{(y - 3)^2}{9} - \frac{(x - 7)^2}{12} = 1$

11. $\frac{(x - 2)^2}{16} + \frac{(y + 1)^2}{7} = 1$

12. $\frac{y^2}{36} + \frac{(x + 6)^2}{20} = 1$

In Exercises 13–20, match the equation with its graph.



$x^2 = 4py$

$x^2 = 12y$

$(x+2)+3 = -4$

12

13. $y^2 = -3x$

14. $\dfrac{(x-2)^2}{4} + y^2 = 1$

15. $\dfrac{y^2}{5} - x^2 = 1$

16. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$

17. $\dfrac{y^2}{3} + x^2 = 1$

18. $x^2 = y$

19. $x^2 = -4y$

20. $y^2 = 6x$

In Exercises 21–28, identify the conic. Then complete the square to write the conic in standard form, and sketch the graph.

21. $x^2 - 6x - y - 3 = 0$

22. $x^2 + 4x + 3y^2 - 5 = 0$

23. $x^2 - y^2 - 2x + 4y - 6 = 0$

24. $x^2 + 2x + 4y - 7 = 0$

25. $y^2 - 6x - 4y - 13 = 0$

26. $3x^2 - 6x - 4y - 9 = 0$

27. $2x^2 - 3y^2 - 12x - 24y + 60 = 0$

28. $12x^2 - 4y^2 - 72x - 16y + 44 = 0$

29. Prove that the parabola with focus $(0, p)$ and directrix $y = -p$ has the equation $x^2 = 4py$.

30. Prove that the equation $y^2 = 4px$ represents a parabola with focus $(p, 0)$ and directrix $x = -p$.

In Exercises 31–36, identify the conic. Solve the equation for y and graph it.

31. $3x^2 - 8xy + 6y^2 - 5x - 5y + 20 = 0$

32. $10x^2 - 8xy + 6y^2 + 8x - 5y - 30 = 0$

33. $3x^2 - 2xy - 5x + 6y - 10 = 0$

34. $5xy - 6y^2 + 10x - 17y + 20 = 0$

35. $-3x^2 + 7xy - 2y^2 - x + 20y - 15 = 0$

36. $-3x^2 + 7xy - 2y^2 - 2x + 3y - 10 = 0$

In Exercises 37–48, find the equation for the conic in standard form.

37. Parabola: vertex $(0, 0)$, focus $(2, 0)$

38. Parabola: vertex $(0, 0)$, opens downward, focal width $= 12$

39. Parabola: vertex $(-3, 3)$, directrix $y = 0$

40. Parabola: vertex $(1, -2)$, opens to the left, focal length $= 2$

41. Ellipse: center $(0, 0)$, foci $(\pm 12, 0)$, vertices $(\pm 13, 0)$

42. Ellipse: center $(0, 0)$, foci $(0, \pm 2)$, vertices $(0, \pm 6)$

43. Ellipse: center $(0, 2)$, semimajor axis $= 3$, one focus is $(2, 2)$.

44. Ellipse: center $(-3, -4)$, semimajor axis $= 4$, one focus is $(0, -4)$.

45. Hyperbola: center $(0, 0)$, foci $(0, \pm 6)$, vertices $(0, \pm 5)$

46. Hyperbola: center $(0, 0)$, vertices $(\pm 2, 0)$, asymptotes $y = \pm 2x$

47. Hyperbola: center $(2, 1)$, vertices $(2 \pm 3, 1)$, one asymptote is $y = (4/3)(x - 2) + 1$

48. Hyperbola: center $(-5, 0)$, one focus is $(-5, 3)$, one vertex is $(-5, 2)$.

In Exercises 49–54, find the equation for the conic in standard form.

49. $x = 5 \cos t,\ y = 2 \sin t,\ 0 \le t \le 2\pi$

50. $x = 4 \sin t,\ y = 6 \cos t,\ 0 \le t \le 4\pi$

51. $x = -2 + \cos t,\ y = 4 + \sin t,\ 2\pi \le t \le 4\pi$

52. $x = 5 + 3 \cos t,\ y = -3 + 3 \sin t,\ -2\pi \le t \le 0$

53. $x = 3 \sec t,\ y = 5 \tan t,\ 0 \le t \le 2\pi$

54. $x = 4 \sec t,\ y = 3 \tan t,\ 0 \le t \le 2\pi$

In Exercises 55–62, identify and graph the conic, and rewrite the equation in Cartesian coordinates.

55. $r = \dfrac{4}{1 + \cos\theta}$

56. $r = \dfrac{5}{1 - \sin\theta}$

57. $r = \dfrac{4}{3 - \cos\theta}$

58. $r = \dfrac{3}{4 + \sin\theta}$

59. $r = \dfrac{35}{2 - 7\sin\theta}$

60. $r = \dfrac{15}{2 + 5\cos\theta}$

61. $r = \dfrac{2}{1 + \cos\theta}$

62. $r = \dfrac{4}{4 - 4\cos\theta}$

In Exercises 63–74, use the points $P(-1, 0, 3)$ and $Q(3, -2, -4)$ and the vectors $\mathbf{v} = \langle -3, 1, -2\rangle$ and $\mathbf{w} = \langle 3, -4, 0\rangle$.

63. Compute the distance from P to Q.

64. Find the midpoint of segment PQ.

65. Compute $\mathbf{v} + \mathbf{w}$.

66. Compute $\mathbf{v} - \mathbf{w}$.

67. Compute $\mathbf{v} \cdot \mathbf{w}$.

68. Compute the magnitude of $\mathbf{v}$.

69. Write the unit vector in the direction of $\mathbf{w}$.

70. Compute $(\mathbf{v} \cdot \mathbf{w})(\mathbf{v} + \mathbf{w})$.

71. Write an equation for the sphere centered at P with radius 4.

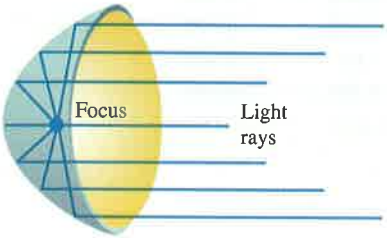
72. Write parametric equations for the line through P and Q.

73. Write a vector equation for the line through P in the direction of $\mathbf{v}$.

74. Write parametric equations for the line in the direction of $\mathbf{w}$ through the midpoint of PQ.

75. Parabolic Microphones B-Ball Network uses a parabolic microphone to capture all the sounds from the basketball players and coaches during each regular season game. If one of its microphones has a parabolic surface generated by the parabola $18y = x^2$, locate the focus (the electronic receiver) of the parabola.

76. Parabolic Headlights Specific Electric makes parabolic headlights for a variety of automobiles. If one of its headlights has a parabolic surface generated by the parabola $y^2 = 15x$ (see figure), where should its lightbulb be placed?



77. Writing to Learn Elliptical Billiard Table Elliptical billiard tables have been constructed with spots marking the foci. Suppose such a table has a major axis of 6 ft and minor axis of 4 ft.

(a) Explain the strategy that a "pool shark" who knows conic geometry would use to hit a blocked spot on this table.

(b) If the table surface is coordinatized so that (0, 0) represents the center of the table and the x-axis is along the focal axis of the ellipse, at which point(s) should the ball be aimed?

78. Weather Satellite The Nimbus weather satellite travels in a north-south circular orbit 500 meters above the Earth. Find the following. (Assume the Earth's radius is 6380 km.)

(a) the velocity of the satellite using the formula for velocity v given for Exercises 43 and 44 in Section 8.5 with $k = 1$

(b) The time required for Nimbus to circle the Earth once

79. Elliptical Orbits The velocity of a body in an elliptical Earth orbit at a distance r (in meters) from the focus (center of the Earth) is

$$v = \sqrt{3.99 \times 10^{14}\left(\frac{2}{r} - \frac{1}{a}\right)}\ \text{m/sec},$$

where a is the semimajor axis of the ellipse. An Earth satellite has a maximum altitude (at *apogee*) of 18,000 km and has a minimum altitude (at *perigee*) of 170 km. Assuming the Earth's radius is 6380 km, find the velocity of the satellite at its apogee and perigee.

80. Icarus The asteroid Icarus is about 1 mi wide. It revolves around the Sun once every 409 Earth days and has an orbital eccentricity of 0.83. Use Kepler's first and third laws to determine Icarus's semimajor axis, perihelion distance, and aphelion distance.

CHAPTER 8 Project

Ellipses as Models of Pendulum Motion

As a simple pendulum swings back and forth, a plot of its velocity with respect to position is elliptical in nature and can be modeled using the standard form of the equation of an ellipse:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$

where x represents the pendulum's a fixed point and y represents the velocity of the pendulum. In this project, you will use a motion detection device to collect distance and velocity data for a swinging pendulum, then find a mathematical model that describes the pendulum's velocity with respect to position.

COLLECTING THE DATA

Construct a simple pendulum by fastening about 0.5 meter of string to a ball. Set up a Calculator-Based Ranger (CBR) system to collect distance and velocity readings for 4 seconds (enough time to capture at least one complete swing of the pendulum). See the CBR guidebook for specific setup instructions. Start the pendulum swinging in front of the detector, then activate the system. The table below is a sample set of data collected in the manner just described.

EXPLORATIONS

1. If you collected data using a CBR, a plot of distance versus time may be shown on your grapher screen. Go to the plot setup screen and create a scatter plot of velocity versus distance. If you do not have access to a CBR, use the distance and velocity data from the table below to create a scatter plot.
2. Find values for a, b, h, and k so that the equation
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$$
fits the velocity versus position data plot. To graph this model you will have to solve the appropriate equation for y and enter it into the calculator in Y1 and Y2.
3. With respect to the ellipse modeled above, what do the variables a, b, h, and k represent?
4. Find the coordinates of the foci of the ellipse modeled above.
5. What is the eccentricity of the ellipse modeled above?
6. Set up plots of distance versus time and velocity versus time. Find models for both of these plots and use them to graph the plot of the ellipse using parametric equations.

Time (sec)	Distance from the CBR (m)	Velocity (m/sec)	Time (sec)	Distance from the CBR (m)	Velocity (m/sec)
0	0.682	−0.3	0.647	0.454	0.279
0.059	0.659	−0.445	0.706	0.476	0.429
0.118	0.629	−0.555	0.765	0.505	0.544
0.176	0.594	−0.621	0.824	0.54	0.616
0.235	0.557	−0.638	0.882	0.576	0.639
0.294	0.521	−0.605	0.941	0.612	0.612
0.353	0.489	−0.523	1	0.645	0.536
0.412	0.463	−0.4	1.059	0.672	0.418
0.471	0.446	−0.246	1.118	0.69	0.266
0.529	0.438	−0.071	1.176	0.699	0.094
0.588	0.442	0.106	1.235	0.698	−0.086